POSITIVE SOLUTIONS OF SINGULAR INTEGRAL EQUATIONS

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ABSTRACT. Continuous, positive solutions of singular integral equations of the form
\[ y(t) = h(t) + \int_0^T k(t,s) [f(y(s)) + g(y(s))] ds \] are sought. Here \( f : [0, \infty) \to [0, \infty) \) is continuous and nondecreasing while \( g : (0, \infty) \to [0, \infty) \) is continuous, nonincreasing and possibly singular. The case when \( T = \infty \) is also discussed.

1. Introduction. In the first half of this paper, Schauder’s fixed point theorem is used to obtain the existence of continuous, positive solutions of
\[ y(t) = h(t) + \int_0^T k(t,s) [f(y(s)) + g(y(s))] ds, \quad t \in [0,T]. \tag{1.1} \]

It is assumed that \( f : [0, \infty) \to [0, \infty) \) is continuous and nondecreasing, while \( g : (0, \infty) \to [0, \infty) \) is continuous, nonincreasing and possibly singular, that is, the possibility of \( g(0) \) being undefined is allowed. In Section 2, by placing appropriate conditions on \( h, k, f \) and \( g \), we use Schauder’s fixed point theorem to prove the existence of a solution \( y \in C[0,T] \) such that \( 0 < \beta < y(t) < \alpha \), \( t \in [0,T] \) for some \( 0 < \beta < \alpha \). In addition a special case of this result, which occurs when \( h \in C[0,T] \) is such that \( h(t) > 0 \), \( t \in [0,T] \), is stated for completeness.

In Section 3 we extend the results of Section 2 and consider the possibly singular equation
\[ y(t) = h(t) + \int_0^\infty k(t,s) [f(y(s)) + g(y(s))] ds, \quad t \in [0,\infty). \tag{1.2} \]

Schauder’s fixed point theorem and the Schauder-Tychonoff fixed point theorem are used to establish the existence of a positive solution \( y \in C[0, \infty) \) and \( y \in BC[0, \infty) \subset C[0, \infty) \) respectively of (1.2). (Here

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