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ON GENERALIZATION OF BULLEN-SIMPSON'S INEQUALITY

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ABSTRACT. Generalization of Bullen-Simpson's inequality for $(2r)$-convex functions is given, by using some Euler type identities. A number of inequalities, for functions whose derivatives are either functions of bounded variation or Lipschitzian functions or functions in L_p-spaces, are proved.

1. Introduction. For any convex function $f : [0,1] \to \mathbf{R}$, the following pair of inequalities, usually referred in the literature as Hadamard's inequalities, hold

$$f\left(\frac{1}{2}\right) \le \int_0^1 f(t)\,dt \le \frac{f(0)+f(1)}{2}. \tag{1.1}$$

If f is concave, the inequalities are reversed. In [**9**] Hammer showed, by a simple geometric argument that for convex functions the absolute value of error in the mid-point quadrature rule is always smaller than absolute value of the error in the trapezoidal rule, i.e., the following inequalities are valid for a convex function f

$$0 \le \int_0^1 f(t)\,dt - f\left(\frac{1}{2}\right) \le \frac{1}{2}\,[f(0)+f(1)] - \int_0^1 f(t)\,dt. \tag{1.2}$$

An elementary analytic proof of (1.1) and (1.2), but stated on the interval $[-1,1]$, was given in [**3**].

The trapezoid rule is the simplest example of a closed quadrature rule, while the mid-point rule is the simplest open quadrature rule, [**4**]. The next simplest such pair is based on the Simpson's formula

$$\int_0^1 f(t)\,dt = \frac{1}{6}\left[f(0)+4f\left(\frac{1}{2}\right)+f(1)\right] - \frac{1}{2880}\,f^{(4)}(\eta) \tag{1.3}$$

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