TAKETA’S THEOREM FOR RELATIVE CHARACTER DEGREES

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Abstract. It has been conjectured by Isaacs that, for finite group $G$, the inequality $dl(N) \leq |cd(G \mid N)|$ holds for all normal solvable subgroups $N$ of $G$. We show that this conjecture holds for $M$-groups. Also, we prove that, if $G$ is solvable and the common-divisor graph $\Gamma(G \mid N)$ is disconnected, then $dl(N) \leq |cd(G \mid N)|$, which is a generalization of [5, Theorem A].

1. Introduction. Let $G$ be a finite group. The idea of this paper is to focus on the relative degree sets, which are certain subsets of $cd(G)$. If $N \triangleleft G$, we define $cd(G \mid N) = \{\chi(1) \mid \chi \in \text{Irr}(G \mid N)\}$, where $\text{Irr}(G \mid N) = \{\chi \in \text{Irr}(G) \mid N \nsubseteq \ker(\chi)\}$. In other words, the relative degree set $cd(G \mid N)$ is the set of degrees of those irreducible characters of $G$ whose kernels do not contain $N$.

Isaacs conjectured that, if $G$ is any finite group, then the inequality $dl(N) \leq |cd(G \mid N)|$ holds for all normal solvable subgroups $N$ of $G$, where $dl(N)$ is the derived length of $N$. In [4], it has been proved by Isaacs and Knutson that this conjecture holds if $G$ is solvable and $|cd(G \mid N)| \leq 3$. Also, in the same paper, they proved that if $N$ is a nilpotent normal subgroup of the finite group $G$, then $dl(N) \leq |cd(G \mid N)|$, and this conjecture holds.

In this paper, we verify this conjecture in two different cases. First, we prove that Isaacs’ conjecture holds for $M$-groups. In other words, we show that, if $G$ is an $M$-group, then the inequality $dl(N) \leq |cd(G \mid N)|$ holds for all normal subgroups $N$ of $G$.

Theorem 1. Let $G$ be an $M$-group. If $N$ is a normal subgroup of $G$, then $dl(N) \leq |cd(G \mid N)|$.

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Note that, if $G$ is any finite group, then $\text{Irr}(G \mid G')$ is exactly the set of nonlinear irreducible characters of $G$, and hence $\text{cd}(G \mid G') = \text{cd}(G) - \{1\}$. It follows from an old result of Taketa when $G$ is an $M$-group that $\text{dl}(G) \leq |\text{cd}(G)|$. Thus, if $G$ is an $M$-group, then the inequality $\text{dl}(N) \leq |\text{cd}(G \mid N)|$ holds for $N = G'$ as $\text{dl}(G') = \text{dl}(G) - 1$. Theorem 1 above implies that the inequality $\text{dl}(N) \leq |\text{cd}(G \mid N)|$ holds not only for $N = G'$ but also for all normal subgroups $N$ of $G$ if $G$ is an $M$-group.

Second, in the last decade, the study of $\text{cd}(G \mid N)$ has been assisted by attaching a graph to $\text{cd}(G \mid N)$. In fact, there are two graphs connected with this set. Let $\rho(\text{cd}(G \mid N))$ be the set of all primes dividing elements of $\text{cd}(G \mid N)$. The prime graph of $\text{cd}(G \mid N)$ is the graph $\Delta(G \mid N)$ with $\rho(\text{cd}(G \mid N))$ as vertices, and there is an edge between $p$ and $q$ if $pq$ divides some element of $\text{cd}(G \mid N)$. The common-divisor graph of $\text{cd}(G \mid N)$ is the graph $\Gamma(G \mid N)$ with $\text{cd}(G \mid N) - \{1\}$ as vertices, and distinct vertices $m$ and $n$ are joined if they have a nontrivial common divisor. It is not difficult to see that these graphs have the same number of connected components. In particular, $\Delta(G \mid N)$ is disconnected if and only if $\Gamma(G \mid N)$ is disconnected.

Isaacs in [3] studied the graph $\Gamma(G \mid N)$. He proved in Theorem A of that paper that, if $N' < N \subseteq G'$, then $\Gamma(G \mid N)$ has at most two connected components. In [7], Lewis removed the hypotheses on $N$, and he proved that $\Gamma(G \mid N)$ has at most three connected components. Also, in [7], he defined a family of groups which he called groups of disconnected type and he could show that a finite group $G$ has a normal subgroup $N$ so that $\Delta(G \mid N)$ is disconnected (and so $\Gamma(G \mid N)$ is disconnected) if and only if either $\Delta(G)$ is disconnected or $G$ is of disconnected type.

Since $\text{cd}(G \mid G') = \text{cd}(G) - \{1\}$, we see that the graph $\Gamma(G \mid G')$ has vertex $\text{cd}(G) - \{1\}$. It follows from [5, Theorem A] that, if $G$ is solvable and the graph $\Gamma(G \mid G')$ is disconnected, then $\text{dl}(G) \leq |\text{cd}(G)|$. As $\text{dl}(G') = \text{dl}(G) - 1$, we deduce that, if the graph $\Gamma(G \mid G')$ is disconnected, then $\text{dl}(G') \leq |\text{cd}(G \mid G')|$. In the following theorem, we let the group $G$ be solvable and we show that the inequality $\text{dl}(N) \leq |\text{cd}(G \mid N)|$ actually holds for all normal subgroups $N$ of $G$ with the property that the graph $\Gamma(G \mid N)$
is disconnected. In other words, we prove that, if $N$ is any normal subgroup of $G$ such that $\Gamma(G \mid N)$ is disconnected, then $\text{dl}(N) \leq |\text{cd}(G \mid N)|$. Not only does this result extend Theorem A of [5], but also we believe that this result will be a tool for studying the long-standing conjecture of Seitz and Isaacs that, if $G$ is a finite solvable group $G$, then $\text{dl}(G) \leq |\text{cd}(G)|$, where $\text{dl}(G)$ is the derived length of $G$.

Theorem 2. Let $G$ be solvable, and let $N \triangleleft G$ be such that the graph $\Gamma(G \mid N)$ is disconnected. Then $\text{dl}(N) \leq |\text{cd}(G \mid N)|$.

2. Proof of Theorem 1. In this section, we prove Theorem 1. Let $G$ be a finite group. Since $\text{Irr}(G \mid G')$ is exactly the set of non-linear irreducible characters of $G$, we have $\text{cd}(G \mid G') = \text{cd}(G) - \{1\}$. It follows from an old result of Taketa, when $G$ is an $M$-group, that $\text{dl}(G') \leq |\text{cd}(G \mid G')|$. This result is Theorem 5.12 of [1]. In Theorem 1, we prove that the inequality $\text{dl}(N) \leq |\text{cd}(G \mid N)|$ holds not only for $N = G'$ but also for all normal subgroups $N$ of $G$, if $G$ is an $M$-group. To do this, we use a similar argument to the standard proof of Taketa’s theorem ([1, Theorem 5.12]).

*Proof of Theorem 1.* Let $f_1 < f_2 < \cdots < f_r$ be the members of $\text{cd}(G \mid N)$. We argue that the $N^{(k)} \subseteq \ker \chi$ if $\chi \in \text{Irr}(G \mid N)$ with $\chi(1) \leq f_k$. This will show that $N^{(r)}$ is contained in the kernel of every member of $\text{Irr}(G \mid N)$. On the other hand, since $N^{(r)} \subseteq N$, this subgroup is also contained in the kernel of every irreducible character of $G$ whose kernel contains $N$. Hence, $N^{(r)}$ is contained in the kernels of all irreducible characters of $G$, and so $N^{(r)} = 1$, and $N$ has derived length at most $r$.

To see this, we use induction on $k$. Let $\chi \in \text{Irr}(G \mid N)$ with $\chi(1) \leq f_k$. Write $\chi = \lambda^G$ for some linear character $\lambda$ of some subgroup $U \leq G$. Non-principal constituents of $(1_U)^G$ have degree less than $\chi(1)$, so each constituent $\psi$ of $(1_U)^G$ either has $N$ in its kernel, or else it lies in $\text{Irr}(G \mid N)$ and has degree at most $f_{k-1}$. In either case, we have $N^{(k-1)} \subseteq \ker \psi$, and thus $N^{(k-1)} \subseteq \ker (1_U)^G \subseteq U$. Then $N^{(k)} \subseteq U' \subseteq \ker \lambda$, and since $N^{(k)}$ is a normal subgroup of $G$, we have $N^{(k)} \subseteq \ker \chi$. This proves the result. (This argument also establishes
the base case of the induction, where \( k = 1 \), because then no constituent of \((1_\U)^G\) lies in \( \text{Irr}(G \mid N) \). \( \square \)

3. Proof of Theorem 2. In this section, we prove Theorem 2. To do this, we need the following lemma which is a consequence of the main theorem of [6].

**Lemma 3.1.** Let \( G \) be a solvable group such that the prime graph \( \Delta(G) \) is disconnected. Then either \( \text{dl}(G) \leq 5 \) or \( G \) satisfies the hypothesis of Example 2.6 of [6].

**Proof.** We suppose that \( G \) does not satisfy the hypothesis of Example 2.6 of [6], and we show that \( \text{dl}(G) \leq 5 \). It follows from the main theorem of [6] that \( G \) is one of the groups included in Examples 2.1–2.5 of [6].

If \( G \) satisfies the hypothesis of Example 2.1 of [6], then \( G \) has a normal nonabelian Sylow \( p \)-subgroup \( P \) and an abelian \( p \)-complement \( K \) for some prime \( p \). It follows from [6, Lemma 3.1] that \( \Delta(G) \) has two connected components, \( \{ p \} \) and \( \pi([G : F]) \), where \( F \) is the Fitting subgroup of \( G \).

We claim that \( K \) fixes every nonlinear irreducible character of \( P \). To do this, let \( \theta \) be a nonlinear irreducible character of \( P \) such that \( K \not\subseteq \text{Stab}_G(\theta) \). This implies that some prime divisor \( l \) of \( |K| \) divides \( |G : \text{Stab}_G(\theta)| \). By applying the Clifford correspondence ([1, Theorem 6.11]), we obtain that \( pl \) divides some character degree of \( G \) as \( \theta(1) \) is a \( p \)-power. Since \( p \neq l \), this is a contradiction because we know that the singleton \( \{ p \} \) is one of the connected components of \( \Delta(G) \). We conclude that \( K \) fixes every nonlinear irreducible character of \( P \) as claimed.

Since \( K \) acts coprimely on \( P \) and \( K \) fixes every nonlinear irreducible character of \( P \), it follows from [2, Theorem 3.3] that \( P' = [P, K]' \) and either \( [P, K] \) is a \( p \)-group of class 2 or \( [P, K] \) is a Frobenius group with kernel \( [P, K]' \). As \( [P, K] \) is a \( p \)-group, we deduce that \( [P, K] \) is not a Frobenius group. Hence, \( [P, K] \) is a \( p \)-group of class 2 and \( \text{dl}([P, K]) = 2 \). We conclude that \( \text{dl}(P) = 2 \) as \( P' = [P, K]' \). Thus, \( \text{dl}(G) \leq 3 \) as \( G/P \) is abelian, which is the desired conclusion.
Suppose that $G$ satisfies the hypothesis of Example 2.2 of [6]. It follows from [6, Lemma 3.2] that $\text{cd} G = \{1, 2, 3, 8\}$, and so $\text{dl} (G) \leq 4$ which is the desired conclusion.

Assume that $G$ satisfies the hypothesis of Example 2.3 of [6]. Then $G$ is the semi-direct product of a subgroup $H$ acting on a subgroup $P_1$ where $P_1$ is an elementary abelian group of order 9 and $\text{cd} H = \{1, 2, 3, 4\}$. This implies that $\text{dl} (H) \leq 4$, and hence $\text{dl} (G) \leq 5$ as $G/P_1$ is isomorphic to $H$. This yields the desired conclusion in this case.

Now, suppose that $F$ is the Fitting subgroup of $G$ and $E/F$ is the Fitting subgroup of $G/F$. If $G$ satisfies the hypothesis of Example 2.4 of [6], then it follows from [6, Lemma 3.4] that $G/F$ is metacyclic and $F = V \times Z$, where $V$ is an elementary abelian group and $Z$ is the central subgroup of $G$. We obtain that $\text{dl} (G/F) \leq 2$ and $\text{dl} (F) = 1$. Thus, $\text{dl} (G) \leq 3$, which is the desired conclusion.

Finally, assume that $G$ satisfies the hypothesis of Example 2.5 of [6]. By applying [6, Lemma 3.5], we deduce that $E$ satisfies the hypotheses of Example 2.1 of [6]. By the second paragraph of the proof, we have that $\text{dl} (E) \leq 3$, and so $\text{dl} (G) \leq 4$ as $G/E$ is cyclic. This completes the proof of the lemma.

Now, we are ready to prove Theorem 2 as a corollary.

**Corollary 3.2.** Let $G$ be solvable, and let $N \triangleleft G$ be such that the graph $\Gamma(G \mid N)$ is disconnected. Then $\text{dl} (N) \leq |\text{cd} (G \mid N)|$.

**Proof.** Since $N \triangleleft G$ is such that the graph $\Gamma(G \mid N)$ is disconnected, it follows from [7, Theorem B] that either $\Gamma(G)$ is disconnected or $G$ is of disconnected type (for the definition of groups of disconnected type, see [7, Definition 4.1]). Suppose that $G$ is of disconnected type but $\Gamma(G)$ is connected. By applying Lemma 4.3 of [7], we determine that $N = [K, P]$, where $K$ is some $2'$-subgroup and $P$ is a normal non abelian 2-subgroup of $G$. Hence, we observe that $N \subseteq P$ as $P$ is normal in $G$. We obtain that $N$ is a normal 2-subgroup of $G$, and so $N$ is a normal nilpotent subgroup of $G$. Corollary 3.3 of [3] implies that $\text{dl} (N) \leq |\text{cd} (G \mid N)|$, which is the desired conclusion.

Therefore, we can assume that $\Gamma(G)$ is also disconnected. Note that, if $N \nsubseteq G'$, then it follows from [7, Lemma 2.1] that $\text{cd} (G) = \text{cd} (G \mid N)$.
We deduce that \( dl(N) \leq dl(G) \leq |cd(G)| = |cd(G \mid N)| \), where the inequality \( dl(G) \leq |cd(G)| \) holds because of Theorem A of [5]. We conclude that \( dl(N) \leq |cd(G \mid N)| \). Thus, we assume that \( N \subseteq G' \).

On the other hand, Theorems B and C of [4] imply that we may assume that \( |cd(G \mid N)| \geq 4 \). Since \( \Gamma(G) \) is disconnected, it follows from Lemma 3.1 that either \( dl(G) \leq 5 \) or \( G \) is the group included in Example 2.6 of [6].

Suppose that \( dl(G) \leq 5 \). Since \( N \subseteq G' \), this implies that \( dl(N) \leq 4 \). We conclude that \( dl(N) \leq |cd(G \mid N)| \) as \( |cd(G \mid N)| \geq 4 \). This is the desired conclusion.

Thus, we assume that \( G \) satisfies the hypotheses of Example 2.6 of [6]. Then [6, Lemma 3.6] implies that \( G/E \) and \( E/F \) are both cyclic, \( G/F' \) satisfies the hypotheses of Example 2.4 of [6], and \( cd(G \mid F') \) consists of degrees that divide \( |P \parallel E : F| \) and are divisible by \( p \mid B \), where \( P \) is a normal Sylow \( p \)-subgroup of \( G \) for some prime \( p \) and \( |B| \) is some divisor of \( |E : F| \). As \( G/F' \) satisfies the hypotheses of [6, Example 2.4], it follows from Lemma 3.4 of [6] that \( cd(G/F' \mid E'/F') = \{|E : F|\} \) as \( F/F' \) is the Fitting subgroup of \( G/F' \).

On the other hand, since \( N \subseteq G' \) and the groups \( G/E \) and \( E/F \) are both cyclic, we see that \( N \subseteq E \) and \( N' \subseteq E' \subseteq F \). This implies that \( cd(G \mid N') \subseteq cd(G \mid E') \). Observe that \( cd(G \mid E') = cd(G \mid F') \cup cd(G/F' \mid E'/F') = cd(G \mid F') \cup \{|E : F|\} \). We determine that \( cd(G \mid N') \subseteq cd(G \mid F') \cup \{|E : F|\} \). We claim that the graph \( \Gamma(G \mid N') \) is connected. To do this, let \( a, b \in cd(G \mid N') \) be arbitrary distinct elements of \( cd(G \mid N') \). Recall that \( cd(G \mid F') \) consists of degrees that divide \( |P \parallel E : F| \) and are divisible by \( p \mid B \), where \( P \) is a normal Sylow \( p \)-subgroup of \( G \) for some prime \( p \) and \( |B| \) is some divisor of \( |E : F| \). As \( cd(G \mid N') \subseteq cd(G \mid F') \cup \{|E : F|\} \), we obtain that either \( a, b \in cd(G \mid F') \) or \( a \in cd(G \mid F') \) and \( b = |E : F| \). If \( a, b \in cd(G \mid F') \), then \( p \) divides both \( a \) and \( b \), and so \( (a, b) > 1 \). If \( a \in cd(G \mid F') \) and \( b = |E : F| \), the \( |B| \) divides both \( a \) and \( b \), and hence \( (a, b) > 1 \). Thus, in both cases, \( a \) and \( b \) are joined. We deduce that the graph \( \Gamma(G \mid N') \) is connected as claimed.

Since the graph \( \Gamma(G \mid N) \) is disconnected, we determine that \( cd(G \mid N') \) is a proper subset of \( cd(G \mid N) \). Also, as \( N' \subseteq F \) is nilpotent, we deduce by Corollary 3.3 of [3] that \( dl(N') \leq |cd(G \mid N')| \). We conclude
that
\[ dl(N) \leq dl(N') + 1 \leq |cd(G \mid N')| + 1 \leq |cd(G \mid N)|, \]
where the last inequality holds because \( cd(G \mid N') \) is a proper set of \( cd(G \mid N) \). This yields the desired conclusion. \( \square \)

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