EXISTENCE OF NONOSCILLATORY SOLUTIONS TO SECOND-ORDER NONLINEAR NEUTRAL DYNAMIC EQUATIONS ON TIME SCALES

JIN GAO AND QIRU WANG

ABSTRACT. By employing Kranoselskii’s fixed point theorem, we establish the existence of nonoscillatory solutions to the second-order nonlinear neutral dynamic equation \( [r(t)(x(t) + p(t)x(g(t)))^\Delta]^\Delta + f(t, x(h(t))) = 0 \) on a time scale. In particular, one interesting example is included to illustrate the versatility of our results.

1. Introduction. Consider second-order nonlinear neutral dynamic equations of the form

\[
[r(t)(x(t) + p(t)x(g(t)))^\Delta]^\Delta + f(t, x(h(t))) = 0
\]

on a time scale \( T \). The motivation originates from [6, 8], where some open problems were presented in [6] and some conditions for the existence of nonoscillatory solutions of first-order nonlinear neutral dynamic equation \( [x(t) + p(t)x(g(t))]^\Delta + f(t, x(h(t))) = 0 \) were presented in [8]. In this paper, by employing Kranoselskii’s fixed point theorem, we try to find some conditions for the existence of nonoscillatory solutions of (1). We remark that there has been a number of researchers studying oscillatory behaviors for dynamic equations on time scales, see, e.g., [1–3, 5–7] and the references therein. However, there are few papers discussing the existence of nonoscillatory solutions for neutral functional dynamic equations on time scales. For a neutral functional dynamic equation, the highest derivative of the unknown function appears with the argument \( t \) (the present state of the system) as well as one or more deviating arguments (the past or future state of the system).
For convenience, we recall some concepts related to time scales. More details can be found in [1, 2].

**Definition 1.** A time scale is an arbitrary nonempty closed subset of the set \( \mathbb{R} \) of real numbers with the topology and ordering inherited from \( \mathbb{R} \). Let \( T \) be a time scale, for \( t \in T \) the forward jump operator is defined by \( \sigma(t) := \inf\{s \in T : s > t\} \), the backward jump operator by \( \rho(t) := \sup\{s \in T : s < t\} \) and the graininess function by \( \mu(t) := \sigma(t) - t \), where \( \inf\emptyset := \sup T \) and \( \sup\emptyset := \inf T \). If \( \sigma(t) > t \), \( t \) is said to be right-scattered; otherwise, it is right-dense. If \( \rho(t) < t \), \( t \) is said to be left-scattered; otherwise, it is left-dense. The set \( T^\kappa \) is defined as follows: if \( T \) has a left-scattered maximum \( m \), then \( T^\kappa = T - \{m\} \); otherwise, \( T^\kappa = T \).

**Definition 2.** For a function \( f : T \to \mathbb{R} \) and \( t \in T^\kappa \), we define the delta-derivative \( f^\Delta(t) \) of \( f(t) \) to be the number (provided it exists) with the property that, given any \( \varepsilon > 0 \), there is a neighborhood \( U \) of \( t \) (i.e., \( U = (t - \delta, t + \delta) \cap T \) for some \( \delta \)) such that

\[
|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s| \quad \text{for all } s \in U.
\]

We say that \( f \) is delta-differentiable (or in short: differentiable) on \( T^\kappa \) provided \( f^\Delta(t) \) exists for all \( t \in T^\kappa \).

It is easily seen that if \( f \) is continuous at \( t \in T \) and \( t \) is right-scattered, then \( f \) is differentiable at \( t \) with

\[
f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.
\]

Moreover, if \( t \) is right-dense, then \( f \) is differential at \( t \) if and only if the limit

\[
\lim_{s \to t} \frac{f(t) - f(s)}{t - s}
\]

exists as a finite number. In this case,

\[
f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.
\]
In addition, if $f^\Delta \geq 0$, then $f$ is nondecreasing.

**Definition 3.** Let $f : T \to \mathbb{R}$ be a function, $f$ is called right-dense continuous (rd-continuous) if it is continuous at right-dense points in $T$ and its left-sided limits exist (finite) at left-dense points in $T$. A function $F : T \to \mathbb{R}$ is called an antiderivative of $f$ provided $F^\Delta(t) = f(t)$ holds for all $t \in T^k$. By the antiderivative, the Cauchy integral of $f$ is defined as $\int_a^b f(s)\Delta s = F(b) - F(a)$, and $\int_a^\infty f(s)\Delta s = \lim_{t \to \infty} \int_a^t f(s)\Delta s$.

Let $C_{rd}(T, \mathbb{R})$ denote the set of all rd-continuous functions mapping $T$ to $\mathbb{R}$. It is shown in [2] that every rd-continuous function has an antiderivative. Since we are interested in the nonoscillatory behavior of (1), we assume throughout that the time scale $T$ under consideration satisfies $\inf T = t_0$ and $\sup T = \infty$.

As usual, by a solution of (1) we mean a function $x(t)$ which is defined on $T$ and satisfies (1) for $t \geq t_0$. A solution $x$ of (1) is said to be eventually positive (or eventually negative) if there exists a $c \in T$ such that $x(t) > 0$ (or $x(t) < 0$) for all $t \geq c$ in $T$. A solution $x$ of (1) is said to be nonoscillatory if it is either eventually positive or eventually negative; otherwise, it is oscillatory.

2. Main results. For $T_0, T_1 \in T$, let $[T_0, \infty)_T := \{t \in T : t \geq T_0\}$ and $[T_0, T_1)_T := \{t \in T : T_0 \leq t \leq T_1\}$. Further, let $C([T_0, \infty)_T, \mathbb{R})$ denote all continuous functions mapping $[T_0, \infty)_T$ into $\mathbb{R}$, and \( (2) \quad BC[T_0, \infty)_T := \{x : x \in C([T_0, \infty)_T, \mathbb{R}) \text{ and } \sup_{t \in [T_0, \infty)_T} |x(t)| < \infty \}. \)

Endowed on $BC[T_0, \infty)_T$ with the norm $||x|| = \sup_{t \in [T_0, \infty)_T} |x(t)|$, $(BC[T_0, \infty)_T, || \cdot ||)$ is a Banach space. Letting $X \subseteq BC[T_0, \infty)_T$, we say that $X$ is uniformly Cauchy if, for any given $\varepsilon > 0$, there exists a $T_1 \in [T_0, \infty)_T$ such that, for any $x \in X$,

$$|x(t_1) - x(t_2)| < \varepsilon \quad \text{for all } t_1, t_2 \in [T_1, \infty)_T.$$ 

$X$ is said to be equicontinuous on $[a, b)_T$ if, for any given $\varepsilon > 0$, there exists a $\delta > 0$ such that for any $x \in X$ and $t_1, t_2 \in [a, b)_T$ with
\[ |t_1 - t_2| < \delta, \quad |x(t_1) - x(t_2)| < \varepsilon. \]

The following is an analogue of the Arzela-Ascoli theorem on time scales.

**Lemma 1** [8, Lemma 4]. Suppose that \( X \subseteq BC[T_0, \infty)_T \) is bounded and uniformly Cauchy. Further, suppose that \( X \) is equicontinuous on \([T_0, T_1]_T \) for any \( T_1 \in [T_0, \infty)_T \). Then \( X \) is relatively compact.

In this section, we will employ Krasnoselskii’s fixed point theorem (see [4]) to establish the existence of nonoscillatory solutions for (1). For the sake of convenience, we state this theorem as follows.

**Lemma 2** (Krasnoselskii’s fixed point theorem). Suppose that \( X \) is a Banach space and \( \Omega \) is a bounded, convex and closed subset of \( X \). Suppose further that two operators \( U, S : \Omega \rightarrow X \) exist such that

(i) \( Ux + Sy \in \Omega \) for all \( x, y \in \Omega \);

(ii) \( U \) is a contraction mapping;

(iii) \( S \) is completely continuous.

Then \( U + S \) has a fixed point in \( \Omega \).

Throughout this section, we will assume in (1) that

(A1) \( r \in C_{rd}(T, (0, \infty)) \) and \( \int_{t_0}^{\infty} 1/r(s) \Delta s < \infty \);

(A2) \( g, h \in C_{rd}(T, T), g(t) \leq t, \lim_{t \to \infty} g(t) = \infty, \lim_{t \to \infty} h(t) = \infty, \) and there exists a \( \{c_k\}_{k \geq 0} \) such that \( \lim_{k \to \infty} c_k = \infty \) and \( g(c_{k+1}) = c_k \);

(A3) \( p \in C_{rd}(T, \mathbb{R}) \) and there exists a constant \( p_0 \) with \( |p_0| < 1 \) such that \( \lim_{t \to \infty} p(t) = p_0 \);

(A4) \( f \in C(T \times \mathbb{R}, \mathbb{R}), f(t, x) \) is nondecreasing in \( x \) and \( x f(t, x) > 0 \) for \( t \in T \) and \( x \neq 0 \).

We note by the assumptions above that, if \( x(t) \) is an eventually negative solution of (1), then \( y(t) = -x(t) \) satisfies

\[ [\Delta r(t)(y(t) + p(t)y(g(t))))^\Delta] \Delta - f(t, -y(h(t)))) = 0. \]
We further note that \( \overline{f}(t, u) := -f(t, -u) \) is nondecreasing in the second variable and \( u \overline{f}(t, u) > 0 \) for \( t \in T \) and \( u \neq 0 \). Hence, in the following, we will restrict our attention to eventually positive solutions of (1).

In the sequel, we use the notation

\begin{equation}
(3) \quad z(t) = x(t) + p(t)x(g(t)).
\end{equation}

Now, we present our first theorem for a classification scheme of the eventually positive solutions to equation (1).

**Theorem 1.** If \( x(t) \) is an eventually positive solution of (1), then either \( \lim_{t \to \infty} x(t) = a > 0 \) or \( \lim_{t \to \infty} x(t) = 0 \).

**Proof.** Suppose that \( x(t) \) is an eventually positive solution of (1). In view of conditions (A2) and (A3), \( T_1 \in T \) and \( |p_0| < p_1 < 1 \) exist such that \( x(h(t)) > 0, x(g(t)) > 0 \) and \( |p(t)| \leq p_1 \) for all \( t \in [T_1, \infty)_T \). Then, from (1) and (3), we have \( [r(t)z^\Delta(t)]^\Delta < 0 \) on \( [T_1, \infty)_T \), which means that \( r(t)z^\Delta(t) \) is decreasing on \( [T_1, \infty)_T \). Then

\[
r(t)z^\Delta(t) \leq r(T_1)z^\Delta(T_1)
\]

or

\begin{equation}
(4) \quad z^\Delta(t) \leq \frac{r(T_1)z^\Delta(T_1)}{r(t)}
\end{equation}

on \( [T_1, \infty)_T \). If there exists a \( t_1 \in T \) with \( t_1 \geq T_1 \) such that \( z^\Delta(t_1) \leq 0 \), then \( r(t)z^\Delta(t) \leq 0 \) for \( t \geq t_1 \) and so \( z^\Delta(t) \) is eventually negative. Otherwise, if, for every \( t \geq T_1 \), \( z^\Delta(t) > 0 \), then \( z^\Delta(t) \) is eventually positive. Hence, \( z(t) \) is always monotonic eventually.

Integrating (4) from \( T_1 \) to \( t(\geq T_1) \), by (A1), we have

\[
z(t) - z(T_1) \leq r(T_1)z^\Delta(T_1) \int_{T_1}^{t} \frac{1}{r(s)} \Delta s
\]

\[
< r(T_1) |z^\Delta(T_1)| \int_{T_1}^{\infty} \frac{1}{r(s)} \Delta s,
\]
which implies that \( z(t) \) is upper bounded.

Now, we claim that \( \lim_{t \to \infty} z(t) \) exists (finite) and is nonnegative. Otherwise, \( \lim_{t \to \infty} z(t) < 0 \) or \( \lim_{t \to \infty} z(t) = -\infty \), which implies that there exists a \( T_2 \geq T_1 \) such that

\[
    z(t) < 0 \text{ or } x(t) < -p(t)x(g(t)) < p_1x(g(t)) \text{ for } t \in [T_2, \infty)_T.
\]

By (A2), we can choose some positive integer \( k_0 \) such that \( c_k \geq T_2 \) for all \( k \geq k_0 \). Then, for any \( k \geq k_0 + 1 \), we have

\[
    x(c_k) < p_1x(g(c_k)) = p_1x(c_{k-1}) < p_1^2x(g(c_{k-1})) = p_1^2x(c_{k-2}) < \cdots < p_1^{k-k_0}x(g(c_{k+1})) = p_1^{k-k_0}x(c_{k_0}).
\]

The inequality above implies that \( \lim_{k \to \infty} x(c_k) = 0 \). It follows from (3) that \( \lim_{k \to \infty} x(c_k) = 0 \) which contradicts \( \lim_{t \to \infty} z(t) < 0 \) or \( \lim_{t \to \infty} z(t) = -\infty \).

Next, we assert that \( x(t) \) is bounded. If this is not true, \( \{t_k\} \) exists with \( t_k \to \infty \) as \( k \to \infty \) such that

\[
    x(t_k) = \max_{t_0 \leq s \leq t_k} x(s) \text{ and } \lim_{k \to \infty} x(t_k) = \infty.
\]

Since \( g(t) \leq t \) and

\[
    z(t_k) = x(t_k) + p(t_k)x(g(t_k)) \geq (1 - |p(t_k)|)x(t_k),
\]

it follows from (A3) that \( \lim_{k \to \infty} z(t_k) = \infty \), which contradicts the conclusion above that \( \lim_{t \to \infty} z(t) = b \geq 0 \) and \( b \) is finite. Hence, \( x(t) \) is bounded.

Finally, we assume that

\[
    \limsup_{t \to \infty} x(t) = \overline{x}, \quad \liminf_{t \to \infty} x(t) = \underline{x}.
\]

If \( 0 \leq p_0 < 1 \), we have

\[
    b \geq \overline{x} + p_0\overline{x} \quad \text{and} \quad b \leq \underline{x} + p_0\overline{x},
\]

which implies that \( \overline{x} \leq \underline{x} \). Thus, \( \overline{x} = \underline{x} \) when \( 0 \leq p_0 < 1 \).
If \(-1 < p_0 < 0\), we have

\[ b \geq \overline{x} + p_0 \overline{x} \quad \text{and} \quad b \leq \underline{x} + p_0 \underline{x}, \]

which implies that \( \overline{x} \leq \underline{x} \). Thus, \( \overline{x} = \underline{x} \) when \(-1 < p_0 < 0\).

To sum up, we see that \( \lim_{t \to \infty} x(t) \) exists and \( \lim_{t \to \infty} x(t) = b/(1 + p_0) \). The proof is complete. \( \Box \)

Next, we will give the existence criteria for each type of solution.

**Theorem 2.** Equation (1) has an eventually positive solution \( x(t) \) with \( \lim_{t \to \infty} x(t) = a > 0 \) if and only if there exists some constant \( K > 0 \) such that

\[
\int_{t_0}^{\infty} \int_{t_0}^{s} \frac{f(t, K)}{r(s)} \Delta t \Delta s < \infty. \tag{5}
\]

**Proof.** Suppose that \( x(t) \) is an eventually positive solution of (1) satisfying \( \lim_{t \to \infty} x(t) = a > 0 \). Then \( \lim_{t \to \infty} z(t) = (1 + p_0)a \), and \( T_1 \in \mathbf{T} \) exists such that \( x(h(t)) \geq a/2 \) for \( t \in [T_1, \infty)_\mathbf{T} \). Integrating (1) from \( T_1 \) to \( s(\geq T_1) \), we have

\[ r(s)z^\Delta(s) - r(T_1)z^\Delta(T_1) = - \int_{T_1}^{s} f(u, x(h(u))) \Delta u, \]

or

\[
z^\Delta(s) = \frac{r(T_1)z^\Delta(T_1)}{r(s)} - \int_{T_1}^{s} \frac{f(u, x(h(u)))}{r(s)} \Delta u. \tag{6}
\]

Integrating (6) from \( T_1 \) to \( t(\geq T_1) \), we obtain

\[
z(t) - z(T_1) = r(T_1)z^\Delta(T_1) \int_{T_1}^{t} \frac{1}{r(s)} \Delta s - \int_{T_1}^{t} \int_{T_1}^{s} \frac{f(u, x(h(u)))}{r(s)} \Delta u \Delta s. \tag{7}
\]

Letting \( t \to \infty \) in (7), we have

\[
\int_{T_1}^{\infty} \int_{T_1}^{s} \frac{f(u, x(h(u)))}{r(s)} \Delta u \Delta s < \infty.
\]
In view of (A4), we have

\[ f\left(u, \frac{a}{2}\right) \leq f(u, x(h(u))), \quad \text{for } u \in [T_1, \infty)_T \]

and

\[ \int_{T_1}^{\infty} \int_{T_1}^{s} \frac{f(u, (a/2))}{r(s)} \Delta u \Delta s \leq \int_{T_1}^{\infty} \int_{T_1}^{s} \frac{f(u, x(h(u)))}{r(s)} \Delta u \Delta s < \infty, \]

which means that (5) holds. The necessary condition is proved.

Conversely, suppose that there exists some constant \( K > 0 \) such that (5) holds. There are two cases to be considered: \( 0 \leq p_0 < 1 \) and \( -1 < p_0 < 0 \).

In the case \( 0 \leq p_0 < 1 \), take \( p_1 \) so that \( p_0 < p_1 < (1 + 4p_0)/5 < 1 \); then \( p_0 > (5p_1 - 1)/4 \).

Since \( \lim_{t \to \infty} p(t) = p_0 \) and (5) holds, we can choose \( T_0 \in T \) large enough such that

\[ \frac{5p_1 - 1}{4} \leq p(t) \leq p_1 < 1, \quad t \in [T_0, \infty)_T \]

and

\[ \int_{T_0}^{\infty} \int_{t_0}^{s} \frac{f(t, K)}{r(s)} \Delta t \Delta s \leq \frac{(1 - p_1)K}{8}. \]

Furthermore, from (A2), we see that \( T_1 \in T \) exists with \( T_1 > T_0 \) such that \( g(t) \geq T_0 \) and \( h(t) \geq T_0 \) for \( t \in [T_1, \infty)_T \).

Define the Banach space \( BC[t_0, \infty)_T \) as in (2), and let

\[ \Omega = \left\{ x = x(t) \in BC[t_0, \infty)_T : \frac{K}{2} \leq x(t) \leq K \right\}. \]

It is easy to verify that \( \Omega \) is a bounded, convex and closed subset of \( BC[t_0, \infty)_T \). By (A4), we have that, for any \( x \in \Omega \),

\[ f(t, x(h(t))) \leq f(t, K), \quad t \in [T_1, \infty)_T. \]
Now we define two operators $S_1$ and $S_2 : \Omega \to BC[t_0, \infty)_T$ as follows:

$$(S_1 x)(t) = \begin{cases} 
(3Kp_1)/4 - p(t)x(g(t)) & t \in [T_1, \infty)_T, \\
(S_1 x)(T_1) & t \in [t_0, T_1]_T, 
\end{cases}$$

and

$$(S_2 x)(t) = \begin{cases} 
(3K/4) + \int_t^\infty \int_t^s [(f(u, x(h(u))))/r(s)]\Delta u \Delta s & t \in [T_1, \infty)_T, \\
(S_2 x)(T_1) & t \in [t_0, T_1]_T. 
\end{cases}$$

(11)

Next, we will show that $S_1$ and $S_2$ satisfy the conditions in Lemma 2.

(i) We first prove that $S_1 x + S_2 y \in \Omega$ for any $x, y \in \Omega$. Note that, for any $x, y \in \Omega$, $K/2 \leq x \leq K$ and $K/2 \leq y \leq K$. For any $x, y \in \Omega$ and $t \in [T_1, \infty)_T$, by (8) and (9), we have:

$$(S_1 x)(t) + (S_2 y)(t) = \frac{3(1 + p_1)K}{4} - p(t)x(g(t)) + \int_t^\infty \int_t^s f(u, x(h(u)))/r(s)\Delta u \Delta s$$

and

$$(S_1 x)(t) + (S_2 y)(t) \leq \frac{3(1 + p_1)K}{4} - p(t)x(g(t)) + \int_t^\infty \int_t^s f(u, K)/r(s)\Delta u \Delta s$$

$$= \frac{3(1 + p_1)K}{4} - 5p_1 - \frac{1}{4} \times \frac{K}{2} + \frac{(1 - p_1)K}{8} = K.$$ 

Similarly, we can prove that $K/2 \leq (S_1 x)(t) + (S_2 y)(t) \leq K$ for any $x, y \in \Omega$ and $t \in [t_0, T_1]_T$. Hence, $S_1 x + S_2 y \in \Omega$ for any $x, y \in \Omega$.

(ii) We prove that $S_1$ is a contraction mapping. Indeed, for $x, y \in \Omega$, we have

$$|(S_1 x)(t) - (S_1 y)(t)| = |p(t)[x(g(T_1)) - y(g(T_1))]|$$

$$\leq p_1 \sup_{t \in [t_0, \infty)_T} |x(t) - y(t)|$$
for \( t \in [t_0, T_1] \) and

\[
| (S_1 x)(t) - (S_1 y)(t) | = | p(t) [(x(g(t)) - y(g(t))] | \\
\leq p_1 \sup_{t \in [t_0, \infty)_T} |x(t) - y(t)|
\]

for \( t \in [T_1, \infty)_T \). Therefore, we have

\[
|| S_1 x - S_1 y || \leq p_1 || x - y ||
\]

for any \( x, y \in \Omega \). Hence, \( S_1 \) is a contraction mapping.

(iii) We will prove that \( S_2 \) is a completely continuous mapping.

First, for \( t \in [t_0, \infty)_T \), we see that \((S_2 x)(t) > 3K/4\) and \((S_2 x)(t) \leq 3K/4 + (1 - p_1)K/8 = (7 - p_1)K/8 < K\). That is, \( S_2 \) maps \( \Omega \) into \( \Omega \).

Second, we consider the continuity of \( S_2 \). Let \( x_n \in \Omega \) and \( ||x_n - x|| \to 0 \) as \( n \to \infty \). Then \( x_n \in \Omega \) and \( |x_n(t) - x(t)| \to 0 \) as \( n \to \infty \) for any \( t \in [t_0, \infty)_T \). Consequently, for \( t \in [T_1, \infty)_T \),

\[
| f(t, x_n(h(t))) - f(t, x(h(t))) | \to 0 \quad \text{as} \quad n \to \infty,
\]

and, for \( t \in T \),

\[
| (S_2 x_k)(t) - (S_2 x)(t) | \\
\leq \int_{T_1}^{\infty} \int_{t_0}^{s} \left| \frac{f(u, x_k(h(u))) - f(u, x(h(u)))}{r(s)} \right| \Delta u \Delta s.
\]

By applying the Lebesgue dominated convergence theorem, we conclude that

\[
||(S_2 x_k)(t) - (S_2 x)(t)|| \to 0 \quad (k \to \infty).
\]

This means that \( S_2 \) is continuous.

Third, we show \( S_2 \Omega \) is relatively compact. According to Lemma 1, it suffices to show that \( S_2 \Omega \) is bounded, uniformly Cauchy and equicontinuous. The boundedness is obvious. Since \( \int_{t_0}^{\infty} \int_{t_0}^{s} [f(t, K)/r(s)] \Delta t \Delta s < \infty \), for any \( \varepsilon > 0 \), there exists a \( T_2 \in [T_1, \infty)_T \) large enough so that \( \int_{T_2}^{\infty} \int_{t_0}^{s} [f(u, K)/r(s)] \Delta u \Delta s < \varepsilon /2 \). Then, for any \( x \in \Omega \) and
\[ t_1, t_2 \in [T_2, \infty)_T, \text{ we have} \]
\[
| (S_2x)(t_1) - (S_2x)(t_2) | = \left| \int_{t_1}^{\infty} \int_{t_0}^{s} \frac{f(u, x(h(u)))}{r(s)} \Delta u \Delta s 
- \int_{t_2}^{\infty} \int_{t_0}^{s} \frac{f(u, x(h(u)))}{r(s)} \Delta u \Delta s \right|
\leq \int_{t_1}^{\infty} \int_{t_0}^{s} \frac{f(u, x(h(u)))}{r(s)} \Delta u \Delta s
+ \int_{t_2}^{\infty} \int_{t_0}^{s} \frac{f(u, x(h(u)))}{r(s)} \Delta u \Delta s
\leq 2 \int_{T_2}^{\infty} \int_{t_0}^{s} \frac{f(u, K)}{r(s)} \Delta u \Delta s < \varepsilon.
\]

Thus, \( S_2 \Omega \) is uniformly Cauchy.

Also, for \( x \in \Omega, t_1, t_2 \in [\min\{T_1 - 1, t_0\}, T_2 + 1]_T \), we have
\[
| (S_2x)(t_1) - (S_2x)(t_2) | \leq \left| \int_{t_1}^{t_2} \int_{t_0}^{s} \frac{f(u, x(h(u)))}{r(s)} \Delta u \Delta s \right|
\leq \left| \int_{t_1}^{t_2} \int_{t_0}^{s} \frac{f(u, K)}{r(s)} \Delta u \Delta s \right|.
\]

Then \( 0 < \delta < 1 \) exists such that \( | (S_2x)(t_1) - (S_2x)(t_2) | < \varepsilon \) if \( |t_2 - t_1| < \delta \).

For any \( x \in \Omega, t_1, t_2 \in [t_0, T_1]_T \), it is easy to see that \( | (S_2x)(t_1) - (S_2x)(t_2) | = 0 < \varepsilon \). This means that \( S_2 \Omega \) is equicontinuous.

It follows from Lemma 1 that \( S_2 \Omega \) is relatively compact, and then \( S_2 \) is completely continuous.

By Lemma 2, \( x \in \Omega \) exists such that \( (S_1 + S_2)x = x \), which means that \( x(t) \) is a solution of (1). In particular, we have
\[
x(t) = \frac{3(1 + p_1)K}{4} - p(t)x(g(t))
+ \int_{t}^{\infty} \int_{t_0}^{s} \frac{f(u, x(h(u)))}{r(s)} \Delta u \Delta s, \quad t \in [T_1, \infty)_T.
\]
Letting \( t \to \infty \), we have \( \lim_{t \to \infty} z(t) = 3(1 + p_1)K/4 \), and then \( \lim_{t \to \infty} x(t) = 3(1 + p_1)K/(4 + 4p_0) > 0 \). The sufficiency holds when \( 0 \leq p_0 < 1 \).
In the case $-1 < p_0 < 0$, take $p_1$ so that $-p_0 < p_1 < (1 - 4p_0)/5 < 1$; then $p_0 < (1 - 5p_1)/4$. Since $\lim_{t \to \infty} p(t) = p_0$ and (5) holds, we can choose $T_0 \in T$ large enough such that (9) holds and

$$\frac{5p_1 - 1}{4} \leq -p(t) < 1, \quad t \in [T_0, \infty)_T.$$ 

Take $T_1 \in T$ with $T_1 > T_0$ so that $g(t) \geq T_0$ and $h(t) \geq T_0$ for $t \in [T_1, \infty)_T$. Similarly, we introduce the Banach space $BC[t_0, \infty)_T$ and its subset $\Omega$ as in (10). Define operator $S_2$ as in (11) and operator $S_1$ on $\Omega$ as follows

$$(S_1x)(t) = \begin{cases} 
-(3Kp_1)/4 - p(t)x(g(t)) & t \in [T_1, \infty)_T, \\
(S_1x)(T_1) & t \in [t_0, T_1]_T.
\end{cases}$$

Next, we prove that $S_1x + S_2y \in \Omega$ for any $x, y \in \Omega$. Indeed, for any $x, y \in \Omega$ and $t \in [T_1, \infty)_T$, we have

$$(S_1x)(t) + (S_2y)(t) = \frac{3(1 - p_1)K}{4} - p(t)x(g(t))$$

$$+ \int_t^\infty \int_{t_0}^s \frac{f(u, y(h(u)))}{r(s)} \Delta u \Delta s$$

$$\geq \frac{3(1 - p_1)K}{4} - p(t)x(g(t))$$

$$\geq \frac{3(1 - p_1)K}{4} + \frac{5p_1 - 1}{4} \times \frac{K}{2}$$

$$= \frac{(5 - p_1)K}{8} > \frac{K}{2}$$

and

$$(S_1x)(t) + (S_2y)(t) \leq \frac{3(1 - p_1)K}{4} - p(t)x(g(t))$$

$$+ \int_t^\infty \int_{t_0}^s \frac{f(u, K)}{r(s)} \Delta u \Delta s$$

$$\leq \frac{3(1 - p_1)K}{4} + p_1K + \frac{(1 - p_1)K}{8}$$

$$= \frac{(7 + p_1)K}{8} < K.$$
That is, \( S_1x + S_2y \in \Omega \) for any \( x, y \in \Omega \).

The following proof is similar to that of the case \( 0 \leq p_0 < 1 \) and is omitted. By Lemma 2, \( x \in \Omega \) exists such that \( (S_1 + S_2)x = x \), which means that \( x(t) \) is a solution of (1). In particular, we have

\[
x(t) = \frac{3(1 - p_1)K}{4} - p(t)x(g(t)) + \int_t^\infty \int_{t_0}^s \frac{f(u, x(h(u)))}{r(s)} \Delta u \Delta s, \quad t \in [T_1, \infty)_T.
\]

Letting \( t \to \infty \), we have \( \lim_{t \to \infty} z(t) = \frac{3(1 - p_1)K}{4} \), and then \( \lim_{t \to \infty} x(t) = \frac{3(1 - p_1)K}{4 + 4p_0} > 0 \). The sufficiency also holds when \(-1 < p_0 < 0\).

The proof is complete. \( \square \)

**Theorem 3.** If \( T_0 \in T \) exists with \( T_0 > 0 \) such that

\[
(12) \quad p(t)e^{-g(t)} \leq -e^{-t}, \quad t \in [T_0, \infty)_T
\]

and

\[
(13) \quad \int_t^\infty \int_{t_0}^s \frac{f(u, x(h(u)))}{r(s)} \Delta u \Delta s \leq \frac{1}{t} + \frac{p(t)}{g(t)}, \quad t \in [T_0, \infty)_T,
\]

then equation (1) has an eventually positive solution \( x(t) \) with \( \lim_{t \to \infty} x(t) = 0 \).

**Proof.** Take \( T_1 \in T \) with \( T_1 > T_0 \) so that \( g(t) \geq T_0 \) and \( h(t) \geq T_0 \) for \( t \in [T_1, \infty)_T \). Define the Banach space \( BC[t_0, \infty)_T \) as in (2). Let

\[
\Omega = \left\{ x \in BC[t_0, \infty)_T : e^{-t} \leq x(t) \leq \frac{1}{t} \text{ for } t \in [T_1, \infty)_T \text{ and } e^{-T_1} \leq x(t) \leq \frac{1}{t} \text{ for } t \in [t_0, T_1]_T \right\};
\]

then \( \Omega \) is a bounded, convex and closed subset of \( BC[t_0, \infty)_T \). Define an operator \( S \) on \( \Omega \) as follows:

\[
(Sx)(t) = \begin{cases} 
-p(T_1)x(g(T_1)) + \int_{T_1}^t \int_{t_0}^s \frac{f(u, x(h(u)))}{r(s)} \Delta u \Delta s & t \in [t_0, T_1]_T, \\
-p(t)x(g(t)) + \int_t^\infty \int_{t_0}^s \frac{f(u, x(h(u)))}{r(s)} \Delta u \Delta s & t \in [T_1, \infty)_T.
\end{cases}
\]
First, we show that $Sx \in \Omega$ for all $x \in \Omega$. Indeed, from (12) and (13), we have for $t \in [T_1, \infty)_{\mathbb{T}}$,

$$
(Sx)(t) = -p(t)x(g(t)) + \int_t^\infty \int_{t_0}^s \frac{f(u, x(h(u)))}{r(s)} \Delta u \Delta s
$$

$$
\leq -p(t)x(g(t)) + \int_t^\infty \int_{t_0}^s \frac{f(u, [1/h(u)])}{r(s)} \Delta u \Delta s
$$

$$
\leq \frac{-p(t)}{g(t)} + \frac{1}{t} + \frac{p(t)}{g(t)} = \frac{1}{t}
$$

and

$$
(Sx)(t) \geq -p(t)x(g(t)) \geq -p(t)e^{-g(t)} \geq e^{-t}.
$$

Also, $e^{-T_1} \leq (Sx)(t) \leq 1/t$ for $t \in [t_0, T_1]_{\mathbb{T}}$. Thus, we have proved that $Sx \in \Omega$ for all $x \in \Omega$. The rest of the proof is similar to Theorem 2 and hence is omitted.

By Lemma 2 with the operator $U = 0$, there exists an $x \in \Omega$ such that $Sx = x$, which means that $x(t)$ is a solution of (1). Note from the definition of $\Omega$, we see that $x(t)$ is eventually positive and $\lim_{t \to \infty} x(t) = 0$. The proof is complete. \(\square\)

**Example.** Let $q > 1$ and $\mathbb{T} = \{q^n : n \in \mathbb{N}_0\}$, where $\mathbb{N}_0$ is the set of nonnegative integers. Consider the following equation

$$
(14) \quad \left[ t^2 \left[ x(t) + \frac{t + 1}{2t} x(\rho(t)) \right] \Delta \right]^\Delta + \frac{x(\sigma(t))}{t^2 \sigma(t)} = 0, \quad t \in \mathbb{T}.
$$

Then $p(t) = (t + 1)/2t$, $g(t) = \rho(t)$, $h(t) = \sigma(t)$, $f(t, x) = x/(t^2 \sigma(t))$ and $r(t) = t^2$. It is easy to see that all the conditions (A1)-(A4) are satisfied. Also, $\int_1^\infty \int_1^s [f(t, K)/r(s)] \Delta t \Delta s < \infty$ for any given $K > 0$. By Theorem 2, equation (14) has an eventually positive solution $x(t)$ with $\lim_{t \to \infty} x(t) = a > 0$.

**Acknowledgments.** The authors are thankful to the referee for valuable comments and suggestions.
REFERENCES


School of Mathematics and Computational Science, Sun Yat-Sen University, Guangzhou, Guangdong 510275, P.R. China
Email address: 306996140@qq.com

School of Mathematics and Computational Science, Sun Yat-Sen University, Guangzhou, Guangdong 510275, P.R. China
Email address: mcswqr@mail.sysu.edu.cn