ON 2- AND 4-DISSECTIONS
FOR SOME INFINITE PRODUCTS

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ABSTRACT. The 2- and 4-dissections of some infinite products are established in this paper. As corollaries of our results, we derive the 4-dissections of some continued fractions appearing in Ramanujan’s notebooks and their reciprocals.

1. Introduction and main results. Throughout this paper, we let $|q| < 1$. We use the standard notation

$$ (a; q)_\infty = \prod_{i=0}^{\infty} (1 - aq^i) $$

and often write

$$ (a_1, a_2, \ldots, a_n; q)_\infty = (a_1; q)_\infty(a_2; q)_\infty \cdots (a_n; q)_\infty. $$

The Ramanujan theta function is defined by

$$ f(a, b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}, $$

where $|ab| < 1$. The function $f(a, b)$ satisfies the well-known Jacobi triple product identity [5]

$$ f(a, b) = (-a, -b, ab; ab)_\infty. $$

A special case of (1.1) is

$$ f(-q) = f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_\infty. $$

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For $n$ positive, we denote $f(-q^n)$ by $f_n$ in this paper for convenience.

Recall that the Rogers-Ramanujan continued fraction is defined by

$$R(q) = 1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \cdots}}} = \frac{(q^2, q^3; q^5)_\infty}{(q, q^4; q^5)_\infty}.$$  

This identity was first established by Rogers [16]. Ramanujan [14] gave 2-dissections of this continued fraction and its reciprocal, and these were first proved by Andrews [2]. Ramanujan [14] also gave 5-dissections of $R(q)$ and its reciprocal, and these results were improved upon and proved by Hirschhorn [8]. In the same paper, Hirschhorn conjectured formulas for 4-dissections of $R(q)$ and its reciprocal, and these were first proved by Lewis and Liu [12]. Hirschhorn [10] also gave an elementary proof of his conjecture.

Gordon’s continued fraction is

$$G(q) = 1 + q + \frac{q^2}{1 + q^2 + 1 + q^5 + 1 + q^7 + \cdots} = \frac{(q^3, q^5; q^8)_\infty}{(q, q^7; q^8)_\infty}.$$  

This identity was established by Gordon [7]. Hirschhorn [9] established 8-dissections of $G(q)$ and its reciprocal, thereby demonstrating the periodicity of the sign of the coefficients in expansions of $G(q)$ and its reciprocal, and in particular that certain coefficients are zero, a phenomenon first observed and shown by Richmond and Szekeres [15]. Alladi and Gordon [1], Andrews and Bressoud [3] and Chan and Yesilyurt [6] generalized these themes. Recently, Xia and Yao [20] proved Hirschhorn’s results by an iterative method.

Ramanujan’s cubic continued fraction is defined by

$$RC(q) = \frac{1}{1+ \frac{q + q^2}{1+ \frac{q^2 + q^4}{1+ \frac{q^3 + q^6}{1+ \cdots}}} = \frac{(q, q^5; q^6)_\infty}{(q^3, q^3; q^6)_\infty}.$$  

This identity was first established by Ramanujan [14]. 2- and 4-dissections of $1/(RC(q))$ were first given by Srivastava [18]. Hirschhorn and Roselin [11] also obtained the 2-, 3-, 4- and 6-dissections of Ramanujan’s cubic continued fraction and its reciprocal.

The objective of this paper is to establish 4-dissections of some infinite products. As corollaries of our results, we obtain 4-dissections of
the reciprocal of Ramanujan-Selberg continued fraction, a continued fraction introduced by Vasuki, Bhaskar and Sharath \[19\], Ramanujan’s cubic continued fraction and its reciprocal. We also discovered the periodicity of the sign of the coefficients in the expansion of the reciprocal of Ramanujan-Selberg continued fraction. Our main results can be stated as follows.

**Theorem 1.1.** Let \( m \) be a positive number and \( u \) an odd number. If \( m \equiv 0 \pmod{4} \), we have

\[
q^{2u}, q^{m-2u}; q^m \infty = (q^u, q^{m-u}; q^m \infty)^{-1}(q^{6m-4u}, -q^{10m+4u}, q^{16m}; q^{16m} \infty)
\]

If \( m \equiv 2 \pmod{4} \), we have

\[
q^{2u}, q^{m-2u}; q^m \infty = (q^u, q^{m-u}; q^m \infty)^{-1}(q^{6m-4u}, -q^{10m-4u}, q^{16m}; q^{16m} \infty)
\]

The Ramanujan-Selberg continued fraction is

\[
S(q) = \frac{1}{1+ \frac{q}{1+ \frac{q^2}{1+ \frac{q^3}{1+ \frac{q^4}{\ddots}}}}} = (q; q^2 \infty) \frac{(q^2, q^4) \infty}{(q^2, q^4) \infty}.
\]
Independently, Ramanujan [13] and Selberg [17] discovered this inter-
esting continued fraction. Setting \( u = 1 \) and \( m = 4 \) in (1.4), we obtain
the 4-dissection of \( 1/(S(q)) \).

**Corollary 1.2.** We have

\[
(1.7) \quad \frac{1}{S(q)} = \frac{(q^2; q^4)_\infty^2}{(q; q^2)_\infty^2}
\]

\[
= \frac{(-q^{20}, -q^{41}; q^{64})_\infty}{(q^4, q^8, q^{16}, q^{24}, q^{28}; q^{32}; q^{64})_\infty (q^{32}; q^{64})_\infty}
\]

\[
+ \frac{q^2(-q^{12}, -q^{52}; q^{64})_\infty}{(q^4, q^8, q^{16}, q^{24}, q^{28}; q^{32})_\infty (q^{32}; q^{64})_\infty}
\]

\[
- \frac{q^7(-q^{12}, -q^{52}; q^{64})_\infty}{(q^8, q^{12}, q^{16}, q^{20}, q^{24}; q^{32})_\infty (q^{32}; q^{64})_\infty}.
\]

From (1.7), we immediately obtain the following corollary on the
periodicity of the signs of the coefficients of \( S^{-1}(q) \):

**Corollary 1.3.** Let

\[
S^{-1}(q) = \frac{(q^2; q^4)_\infty^2}{(q; q^2)_\infty} = \sum_{n=0}^{\infty} d_n q^n.
\]

For \( n \geq 0 \), we have

\[
(1.8) \quad d_{4n} \geq 0, \quad d_{4n+1} \geq 0, \quad d_{4n+2} \leq 0, \quad d_{4n+7} \leq 0.
\]

Vasuki, Bhaskar and Sharath [19] introduced the following continued
fraction

\[
X(q) = \frac{(q, q^5; q^6)_\infty}{(q^2, q^4; q^6)_\infty}
\]

\[
= \frac{q^{1/2}(1-q^2)}{(1-q^{3/2}) + \frac{(1-q^{1/2})(1-q^{7/2})}{(1-q^{5/2})(1-q^{13/2})}} + \ldots.
\]
Taking $u = 1$ and $m = 6$ in (1.5), we obtain the 4-dissection of $1/(X(q))$.

**Corollary 1.4.** We have

\[
(1.9) \quad \frac{(q^2, q^4; q^6)_\infty}{(q, q^5; q^6)_\infty} = \frac{(q^4, q^8; q^{12})_\infty}{(q^{12}, q^{24}; q^{24})_\infty} (-q^8, q^{16}; q^{24})_\infty \frac{(-q^{36}, q^{60}, q^{96}; q^{96})_\infty}{(q^{12}, q^{24}; q^{24})_\infty}
\]

\[
+ q \frac{(q^4, q^8; q^{12})_\infty}{(q^{12}, q^{24}; q^{24})_\infty} (-q^4, q^{20}; q^{24})_\infty \frac{(-q^{36}, q^{60}, q^{96}; q^{96})_\infty}{(q^{12}, q^{24}; q^{24})_\infty}
\]

\[
+ q^6 \frac{(q^4, q^8; q^{12})_\infty}{(q^{12}, q^{24}; q^{24})_\infty} (-q^8, q^{16}; q^{24})_\infty \frac{(-q^{12}, q^{84}, q^{96}; q^{96})_\infty}{(q^{12}, q^{24}; q^{24})_\infty}
\]

\[
+ q^7 \frac{(q^4, q^8; q^{12})_\infty}{(q^{12}, q^{24}; q^{24})_\infty} (-q^4, q^{20}; q^{24})_\infty \frac{(-q^{12}, q^{84}, q^{96}; q^{96})_\infty}{(q^{12}, q^{24}; q^{24})_\infty}.
\]

**Theorem 1.5.** Let $m$ and $u$ be positive integers. If $m \equiv 2 \pmod{4}$ and $u$ is odd, we have

\[
(1.10) \quad \frac{(q^{3u}, q^{m-3u}; q^m)_\infty}{(q^u, q^{m-u}; q^m)_\infty} = \frac{(q^{2m-4u}; q^{2m+4u}; q^{4m})_\infty}{(q^{2m}; q^{4m})_\infty} \frac{(q^{m-2u}; q^{m+2u}; q^{2m})_\infty}{(q^{2m}; q^{4m})_\infty}
\]

\[
\times \left((-q^{m+2u}, q^{3m-2u}; q^{4m})_\infty^2 + q^{2u} (-q^{m-2u}, q^{3m+2u}; q^{4m})_\infty^2\right)
\]

\[
+ q^u \frac{(q^{2m}; q^{4m})_\infty^4}{(q^{2m}; q^{4m})_\infty} \left(\left(\frac{q^{2m-4u}; q^{2m+4u}; q^{4m}}{q^{2m}; q^{4m}}\right)_\infty^2 - q^{m-4u} \left(\frac{q^{4u}; q^{4m-4u}; q^{4m}}{q^{4u}; q^{4m}}\right)_\infty^2\right).
\]

Letting $u = 1$ and $m = 6$ in (1.10), we obtain the 4-dissection of the reciprocal of the Ramanujan cubic continued fraction.

**Corollary 1.6.** We have

\[
(1.11) \quad \frac{1}{RC(q)} = \frac{(q^3, q^6)^2_\infty}{(q, q^5; q^6)_\infty} = \frac{(q^4, q^{20}; q^{24})_\infty}{(q^{12}; q^{24})_\infty^4} \frac{(q^{16}, q^{32}; q^{48})_\infty^2}{(q^{12}; q^{24})_\infty^4} + \frac{(q^8, q^{16}; q^{24})_\infty^2}{(q^{12}; q^{24})_\infty^4}
\]
\[ + q^2 \frac{(q^8, q^{16}, -q^4, -q^{20}, -q^4, -q^{20}; q^{24})_\infty (q^4, q^8; q^{12})_\infty}{(q^{12}; q^{24})_\infty} \]
\[ - q^3 \frac{(q^4, q^{20}; q^{24})_\infty^2}{(q^{12}; q^{24})_\infty^4}. \]

**Theorem 1.7.** Let \( m \) and \( u \) be positive integers. If \( m \equiv 2 \pmod{4} \) and \( u \) is odd, we have

\[ (1.12) \quad \frac{(q^u, q^{m-u}; q^m)_\infty}{(q^{3u}, q^{m-3u}; q^m)_\infty} = \frac{(q^4, q^{2m-4u}, q^{m+2u}, q^{m-2u}, q^{2m}; q^m)_\infty}{(q^{12}, q^{4m-12u}; q^{4m})_\infty^4} \]
\[ + q^6 q^4 u \frac{(q^4, q^{2m-4u}, q^{m+2u}, q^{m-2u}; q^{2m})_\infty}{(q^{12}, q^{4m-12u}; q^{4m})_\infty^4} \]
\[ - q^u \frac{(-q^{m-2u}, -q^{3m+2u}; q^{4m})_\infty}{(q^{12}, q^{4m-12u}; q^{4m})_\infty^4} \]
\[ - q^8 u \frac{(-q^{m+2u}, -q^{3m-2u}; q^{4m})_\infty}{(q^{12}, q^{4m-12u}; q^{4m})_\infty^4} \]
\[ + q^3 u \frac{(-q^{m-6u}, -q^{m+6u}, q^4, q^{2m-4u}, q^{m-2u}, q^{m+2u}; q^{2m})_\infty}{(q^{12}, q^{4m-12u}; q^{4m})_\infty^4}. \]

Taking \( u = 1 \) and \( m = 6 \) in (1.12), we obtain the 4-dissection of \( RC(q) \).

**Corollary 1.8.**

\[ (1.13) \quad \frac{(q, q^5; q^6)_\infty}{(q^3, q^6)_\infty^2} = \frac{(q^4, q^{14})_\infty^2 (q^{24}, q^{24})_\infty^{14}}{(q^{12}, q^{12})_\infty^8 (q^{18}, q^{18})_\infty^4} \]
\[ - q \left( \frac{(-q^{12}, -q^{20}; q^{24})_\infty (-q^{12}; q^{24})_\infty^6}{(q^{12}; q^{24})_\infty^8} \right) \]
\[ - 8q^8 \left( \frac{(-q^8, -q^{16}; q^{24})_\infty (-q^{24}; q^{24})_\infty^6}{(q^{12}; q^{24})_\infty^8} \right). \]
\[ + 4q^6 \frac{(q^4; q^4)^2_\infty (q^{24}; q^{24})^2_\infty (q^{48}; q^{48})^4_\infty}{(q^{12}; q^{12})^8_\infty} + 2q^3 \frac{(q^4; q^4)^2_\infty (q^{24}; q^{24})^8_\infty}{(q^{12}; q^{12})^{10}_\infty}. \]

Note that Hirschhorn and Roselin [11] also derived the following 4-dissection of \( RC(q) \):

\[ (1.14) \]
\[ \frac{(q, q^5; q^6)_\infty}{(q^3; q^6)^2_\infty} = \frac{(q^4; q^4)^2_\infty (q^{24}; q^{24})^{14}_\infty}{(q^{12}; q^{12})^4_\infty (q^{48}; q^{48})^2_\infty} - q \left( \frac{(q^8; q^8)^4_\infty (q^{24}; q^{24})^4_\infty}{(q^{12}; q^{12})^8_\infty} + 4 \frac{(q^4; q^4)^4_\infty (q^{24}; q^{24})^4_\infty}{(q^{8}; q^8)^4_\infty (q^{12}; q^{12})^4_\infty} \right) + 4q^6 \frac{(q^4; q^4)^2_\infty (q^{24}; q^{24})^2_\infty (q^{48}; q^{48})^4_\infty}{(q^{12}; q^{12})^8_\infty} + 2q^3 \frac{(q^4; q^4)^2_\infty (q^{24}; q^{24})^8_\infty}{(q^{12}; q^{12})^{10}_\infty}, \]

which is different from (1.13).

2. Proof of Theorem 1.1. In order to prove our main theorems, we will require the following three identities, which were proved by Berndt [4, pages 45–47], respectively.

Lemma 2.1. We have

\[ (2.1) \]
\[ f(a, b) = f(a^3b, ab^3) + af(b/a, a^5b^3), \]

\[ (2.2) \]
\[ f(a, b)f(c, d) = f(ac, bd)f(ad, bc) + af(b/c, ac^2d)f(b/d, acd^2), \]

\[ (2.3) \]
\[ f(a, b)f(c, d)f(an, b/n)f(cn, d/n) \]
\[ - f(-a, -b)f(-c, -d)f(-an, -b/n)f(-cn, -d/n) \]
\[ = 2af(c/a, ad)f(d/(an), acn)f(n, ab/n)f(ab, a^3b^3), \]

where \( ab = cd \).

Now we are ready to prove Theorem 1.1.
Proof. We first establish the 2-dissection of \[((q^{2u}, q^{m-2u}; q^m)_\infty)/(q^u, q^{m-u}; q^m)_\infty\). Taking \(a = q^u\) and \(b = q^{m-u}\) in (2.1), we have

\begin{equation}
(2.4) 
\frac{f(q^u, q^{m-u})}{f(q^{m+2u}, q^{3m-2u}) + q^u f(q^{m-2u}, q^{3m+2u})}.
\end{equation}

By (2.4), we can derive the 2-dissection of \((q^{2u}, q^{m-2u}; q^m)_\infty/(q^u, q^{m-u}; q^m)_\infty\). It is a routine to verify that

\begin{equation}
(2.5) 
\frac{(q^{2u}, q^{m-2u}; q^m)_\infty}{(q^u, q^{m-u}; q^m)_\infty} = \frac{f_2m f(-q^{2u}, -q^{m-2u}) f(q^u, q^{m-u})}{f_2m f(-q^{2u}, -q^{m-2u})} 
= \frac{f_2m f(-q^{2u}, -q^{m-2u})}{f_2m f(-q^{2u}, -q^{m-2u})} \left( f(q^{m+2u}, q^{3m-2u}) + q^u f(q^{m-2u}, q^{3m+2u}) \right) 
= A(q^2) + q^u B(q^2),
\end{equation}

where

\begin{equation}
A(q) = \frac{(q^{m/2-u}, q^{m/2+u}; q^m)_{\infty}(-q^{m/2-u}, -q^{3m/2-u}, q^{2m}; q^{2m})_{\infty}}{(q^{m/2}; q^{m/2})_{\infty}}
\end{equation}

and

\begin{equation}
B(q) = \frac{(q^{m/2-u}, q^{m/2+u}; q^m)_{\infty}(-q^{m/2-u}, -q^{3m/2+u}, q^{2m}; q^{2m})_{\infty}}{(q^{m/2}; q^{m/2})_{\infty}}.
\end{equation}

We first consider the case \(m \equiv 0 \pmod{4}\). In (2.1), let \(a = -q^{m/2-u}\) and \(b = -q^{3m/2+u}\), we see that

\begin{equation}
(2.6) 
f(-q^{m/2-u}, -q^{3m/2+u}) = f(q^{3m-2u}, q^{5m+2u}) - q^{m/2-u} f(q^{m+2u}, q^{7m-2u}).
\end{equation}

From (2.6), it is easy to verify that

\begin{equation}
(2.7) 
A(q) = \frac{f(-q^{m+2u}, -q^{3m-2u})}{f_{4m} f_{m/2}} f(-q^{m/2-u}, -q^{3m/2+u}) 
= \frac{f(-q^{m+2u}, -q^{3m-2u})}{f_{4m} f_{m/2}} \left( f(q^{3m-2u}, q^{5m+2u}) - q^{m/2-u} f(q^{m+2u}, q^{7m-2u}) \right).
\end{equation}
On the other hand, in (2.6), replacing \(u\) by \(-u\), we have

\[
(2.8) \quad B(q) = \frac{f(-q^{m-2}u, -q^{3m+2}u)}{f_4m f_{m/2}} f(-q^{m/2+u}, -q^{3m/2-u}) \\
= \frac{f(-q^{m-2}u, -q^{3m+2}u)}{f_4m f_{m/2}} \left( f(q^{3m+2}u, q^{5m-2}u) \\
- q^{m/2+u} f(q^{m-2}u, q^{7m+2}u) \right).
\]

In view of (2.5), (2.7) and (2.8), we obtain (1.4).

It remains to consider the case \(m \equiv 2 \pmod{4}\). In (2.1), taking \(a = q^{m/2}\) and \(b = q^{3m/2}\), we have

\[
(2.9) \quad f(q^{m/2}, q^{3m/2}) = f(q^{3m}, q^{5m}) + q^{m/2} f(q^m, q^{7m}).
\]

It follows from (2.9) that

\[
(2.10) \quad A(q) = \frac{f(-q^{m/2-u}, -q^{m/2+u}) f(q^{m/2+u}, q^{3m/2-u})}{f^3_m} f(q^{m/2}, q^{3m/2}) \\
= \frac{f(-q^{m/2-u}, -q^{m/2+u}) f(q^{m/2+u}, q^{3m/2-u})}{f^3_m} \\
\times \left( f(q^{3m}, q^{5m}) + q^{m/2} f(q^m, q^{7m}) \right).
\]

Similarly, by (2.9), we have

\[
(2.11) \quad B(q) = \frac{f(-q^{m/2-u}, -q^{m/2+u}) f(q^{m/2-u}, q^{3m/2+u})}{f^3_m} f(q^{m/2}, q^{3m/2}) \\
= \frac{f(-q^{m/2-u}, -q^{m/2+u}) f(q^{m/2-u}, q^{3m/2+u})}{f^3_m} \\
\times \left( f(q^{3m}, q^{5m}) + q^{m/2} f(q^m, q^{7m}) \right).
\]

Formula (1.5) follows from (2.5), (2.10) and (2.11). The proof is complete. □

3. Proof of Theorem 1.5. In this section, we turn to prove Theorem 1.5.
Proof. We also first derive the 2-dissection of \((q^{3u}, q^{m-3u}; q^m)_\infty/(q^u, q^{m-u}; q^m)_\infty\). In (2.2), letting \(a = q^u\), \(b = q^{m-u}\), \(c = -q^{3u}\) and \(d = -q^{m-3u}\), we have
\[
(3.1) \quad f(q^u, q^{m-u})f(-q^{3u}, -q^{m-3u})
\]
\[
= f(-q^{4u}, -q^{2m-4u})f(-q^{m-2u}, -q^{m+2u})
+ q^uf(-q^{m-4u}, q^{m+4u})f(-q^{2u}, -q^{2m-2u}).
\]
By (3.1), it is easy to verify that
\[
(3.2) \quad \frac{(q^{3u}, q^{m-3u}; q^m)_\infty}{(q^u, q^{m-u}; q^m)_\infty} = \frac{f_{2m}f(-q^{3u}, -q^{m-3u})f(q^u, q^{m-u})}{f_{2m}^2f(-q^{2u}, -q^{2m-2u})}
\]
\[
= \frac{f_{2m}}{f_{2m}^2}\left(\frac{f(-q^{4u}, -q^{2m-4u})f(-q^{m-2u}, -q^{m+2u})}{f(-q^{2u}, -q^{2m-2u})}
+ q^uf(-q^{m-4u}, -q^{m+4u})\right)
\]
\[
= C(q^2) + q^uD(q^2),
\]
where
\[
C(q) = \frac{(q^{2u}, q^{m-2u}, q^{m/2-u}, q^{m/2+u}; q^m)_\infty}{(q^{m/2}, q^{m/2}, q^u, q^{m-u}; q^m)_\infty}
\]
and
\[
D(q) = \frac{(q^{m/2-2u}, q^{m/2+2u}; q^m)_\infty}{(q^{m/2}, q^{m/2}; q^m)_\infty}.
\]
Taking \(a = q^u\), \(b = q^{m-u}\), \(c = q^{m/2}\) and \(d = q^{m/2}\) in (2.2), we obtain
\[
(3.3) \quad f(q^u, q^{m-u})f(q^{m/2}, q^{m/2})
\]
\[
= f^2(q^{m/2+u}, q^{m/2+u}) + q^uf^2(q^{m/2-u}, q^{m/2+u}).
\]
Therefore, by (3.3), we have
\[
(3.4) \quad C(q)
\]
\[
= \frac{f_{2m}f(-q^{2u}, -q^{m-2u})f(-q^{m/2-u}, -q^{m/2+u})}{f_{2m}^2f(-q^{2u}, -q^{2m-2u})}
\]
\[
\times f(q^{m/2}, q^{m/2})f(q^u, q^{m-u})
\]
\[
= \frac{f_{2m}f(-q^{2u}, -q^{m-2u})f(-q^{(m/2)-u}, -q^{(m/2)+u})}{f_{2m}^2f(-q^{2u}, -q^{2m-2u})}
\]
\[
\times \left(\frac{f^2(q^{(m/2)+u}, q^{(m/2)-u}) + q^uf^2(q^{(m/2)-u}, q^{(m/2)+u})}{f^2(q^{(m/2)+u}, q^{(m/2)-u}) + q^uf^2(q^{(m/2)-u}, q^{(m/2)+u})}\right).
\]
Similarly, letting \( a = -q^{m/2-2u}, b = -q^{m/2+2u}, c = q^{m/2} \) and \( d = q^{m/2} \) in (2.2), we see that

\[
(3.5) \quad f(-q^{(m/2)-2u}, -q^{(m/2)+2u}) f(q^{m/2}, q^{m/2}) = f^2(-q^{m-2u}, -q^{m+2u}) - q^{(m/2)-2u} f^2(-q^{2u}, -q^{2m-2u}),
\]

which implies that

\[
(3.6) \quad D(q) = \frac{(q^{m/2-2u}, q^{m/2+2u}, -q^{m/2}, -q^{m/2}; q^{m})_\infty}{(q^{m}, q^{m}; q^{2m})_\infty} = \frac{f_{2m}^2}{f_m^4} f(q^{m/2}, q^{m/2}) f(-q^{m/2-2u}, -q^{m/2+2u}) = \frac{f_{2m}^2}{f_m^4} \left( f^2(-q^{m-2u}, -q^{m+2u}) - q^{m/2-2u} f^2(-q^{2u}, -q^{2m-2u}) \right).
\]

Thus, combining (3.2), (3.4) and (3.6), we obtain (1.10). This completes the proof. \( \Box \)

4. Proof of Theorem 1.7. Now we turn to prove Theorem 1.7.

Proof. We first establish 2-dissection of \((q^{u}, q^{m-u}; q^{m})_\infty/(q^{3u}, q^{m-3u}; q^{m})_\infty\). Taking \( a = -q^u, b = -q^{m-u}, c = q^{3u} \) and \( d = q^{m-3u} \) in (2.2), we have

\[
(4.1) \quad f(-q^u, -q^{m-u}) f(q^{3u}, q^{m-3u}) = f(-q^{4u}, -q^{2m-4u}) f(-q^{m-2u}, -q^{m+2u}) - q^{u} f(-q^{m-4u}, -q^{m+4u}) f(-q^{2u}, -q^{2m-2u}).
\]

In view of (4.1), we see that

\[
(4.2) \quad \frac{(q^{u}, q^{m-u}; q^{m})_\infty}{(q^{3u}, q^{m-3u}; q^{m})_\infty} = \frac{f_{2m} f(-q^u, -q^{m-u}) f(q^{3u}, q^{m-3u})}{f_m^2 f(-q^{6u}, -q^{2m-6u})} = \frac{f_{2m} (f(-q^{4u}, -q^{2m-4u}) f(-q^{m-2u}, -q^{m+2u}) - q^{u} f(-q^{m-4u}, -q^{m+4u}) f(-q^{2u}, -q^{2m-2u}))}{f_m^2 f(-q^{6u}, -q^{2m-6u})}.
\]
Let
\[ E(q) = \frac{f_m f(-q^{2u}, -q^{m-2u}) f(-q^{m/2-2u}, -q^{m/2+u})}{f_{m/2}^2 (-q^{3u}, -q^{m-3u})} \]
and
\[ F(q) = \frac{f_m f(-q^{m/2-2u}, -q^{m/2+2u}) f(-q^{u}, -q^{m-u})}{f_{m/2}^2 (-q^{3u}, -q^{m-3u})}. \]

Hence, by (4.2), we see that
\[ \frac{(q^u, q^{m-u}, q^m)_\infty}{(q^{3u}, q^{m-3u}; q^m)_\infty} = E(q^2) - q^u F(q^2). \]

Taking \( a = q^{3u} \), \( b = q^{m-3u} \), \( c = q^{m/2} \) and \( d = q^{m/2} \) in (2.2), we have
\[ f(q^{m/2}, q^{m/2}) f(q^{3u}, q^{m-3u}) = f^2(q^{m/2+3u}, q^{m/2-3u}) + q^{3u} f^2(q^{m/2-3u}, q^{3m/2+3u}). \]

Therefore, we have
\[ E(q) = \frac{f^3_{2m} f(-q^{2u}, -q^{m-2u}) f(-q^{m/2-2u}, -q^{m/2+u})}{f^6_m f(-q^{6u}, -q^{2m-6u})} \times f(q^{m/2}, q^{m/2}) f(q^{3u}, q^{m-3u}) \]
\[ = \frac{f^3_{2m} f(-q^{2u}, -q^{m-2u}) f(-q^{m/2-2u}, -q^{m/2+u})}{f^6_m f(-q^{6u}, -q^{2m-6u})} \times \left( f^2(q^{m/2+3u}, q^{m/2-3u}) + q^{3u} f^2(q^{m/2-3u}, q^{3m/2+3u}) \right). \]

Setting \( a = -q^u \), \( b = -q^{m-u} \), \( c = -q^{m/2-2u} \), \( d = -q^{m/2+2u} \) and \( n = -q^{m/2-u} \) in (2.3), we have
\[ f(-q^u, -q^{m-u}) f(-q^{m/2-2u}, -q^{m/2+2u}) f(q^{m/2}, q^{m/2}) f(q^{3u}, q^{m-3u}) \]
\[ - f(q^u, q^{m-u}) f(q^{m/2-2u}, q^{m/2+2u}) f(-q^{m/2}, -q^{m/2}) f(-q^{3u}, -q^{m-3u}) \]
\[ = -2q^u f(q^{m/2-3u}, q^{m/2+3u}) f(-q^{2u}, -q^{m-2u}) \times f(-q^{m/2-u}, -q^{m/2+u}) f(q^m, q^{3m}). \]
Taking \( a = -q^u, b = -q^{m-u}, \) \( c = -q^{m/2-u} \) and \( d = -q^{m/2+u} \) in (2.2), we have

\[
(4.7) \quad f(-q^u, -q^{m-u}) f(-q^{m/2-2u}, -q^{m/2+2u}) = f(q^{m/2-u}, q^{3m/2+u}) f(q^{m/2+3u}, q^{3m/2-3u})
\]

\[
- q^u f(q^{m/2+u}, q^{3m/2-2u}) f(q^{m/2-3u}, q^{3m/2+3u}).
\]

In view of (4.4) and (4.7), we have

\[
(4.8) \quad f(-q^u, -q^{m-u}) f(-q^{m/2-2u}, -q^{m/2+2u}) f(q^{m/2}, q^{m/2}) f(q^{3u}, q^{m-3u})
\]

\[
+ f(q^u, q^{m-u}) f(q^{m/2-2u}, q^{m/2+2u})
\]

\[
\times f(-q^{m/2}, -q^{m/2}) f(-q^{3u}, -q^{m-3u})
\]

\[
= 2 f(q^{m/2-u}, q^{3m/2+u}) f^3(q^{m/2}+3u, q^{3m/2}-3u)
\]

\[
- 2 q^4 f(q^{m/2}+u, q^{3m/2}-u) f^3(q^{m/2}-3u, q^{3m/2}+3u).
\]

It follows from (4.6) and (4.8) that

\[
(4.9) \quad f(-q^u, -q^{m-u}) f(-q^{m/2-2u}, -q^{m/2+2u}) f(q^{m/2}, q^{m/2}) f(q^{3u}, q^{m-3u})
\]

\[
= f(q^{m/2-u}, q^{3m/2+u}) f^3(q^{m/2}+3u, q^{3m/2}-3u)
\]

\[
- q^4 f(q^{m/2}+u, q^{3m/2}-u) f^3(q^{m/2}-3u, q^{3m/2}+3u)
\]

\[
- q^u f(q^{m/2-3u}, q^{m/2+3u}) f(-q^{2u}, -q^{m-2u})
\]

\[
\times f(-q^{m/2-u}, -q^{m/2+u}) f(q^m, q^{3m}).
\]

Therefore, we have

\[
(4.10) \quad F(q) = f_{2m}^3 f(-q^u, -q^{m-u}) f(-q^{m/2-2u}, -q^{m/2+2u}) f(q^{m/2}, q^{m/2}) f(q^{3u}, q^{m-3u})
\]

\[
- q^4 f(q^{m/2}+u, q^{3m/2}-u) f^3(q^{m/2}-3u, q^{3m/2}+3u)
\]

\[
- q^u f(q^{m/2-3u}, q^{m/2+3u}) f(-q^{2u}, -q^{m-2u})
\]

\[
\times f(-q^{m/2-u}, -q^{m/2+u}) f(q^m, q^{3m}).
\]
It follows from (4.2), (4.5) and (4.10) that

\[
\frac{(q,u,q^{m-u},q^{m})_{\infty}}{(q^{3},q^{m-4u},q^{3m})_{\infty}} = \frac{f_{4m}^{3}f(-q^{4u},-q^{2m-4u})f(-q^{m-2u},-q^{m+2u})}{f_{2m}^{2}f(-q^{12u},-q^{4m-12u})} f^{2}(q^{m+6u},q^{3m-6u}) \\
+ q^{6u} \frac{f_{4m}^{3}f(-q^{4u},-q^{2m-4u})f(-q^{m-2u},-q^{m+2u})}{f_{2m}^{2}f(-q^{12u},-q^{4m-12u})} \\
\times f^{2}(q^{m-6u},q^{3m+6u}) \\
- q^{u} \left( \frac{f_{4m}^{3}f(q^{m-2u},q^{3m+2u})f^{3}(q^{m+6u},q^{3m-6u})}{f_{2m}^{2}f(-q^{12u},-q^{4m-12u})} \right) \\
- q^{8u} \frac{f(q^{m+2u},q^{3m-2u})f^{3}(q^{m-6u},q^{3m+6u})}{f_{2m}^{2}f(-q^{12u},-q^{4m-12u})} \\
+ q^{3u} \frac{f_{4m}^{3}f(q^{m-6u},q^{m+6u})f(-q^{4u},-q^{2m-4u})}{f_{2m}^{2}f(-q^{12u},-q^{4m-12u})} \\
\times f(-q^{m-2u},-q^{m+2u})f(q^{m},q^{6m}) \\
\times f_{2m}^{2}f(-q^{12u},-q^{4m-12u}),
\]

which is nothing but (1.12). The proof is complete. □

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