ABSTRACT. We prove the following sub- and super-additive properties of the psi function.

(i) The inequality
\[ \psi((x+y)^\alpha) \leq \psi(x^\alpha) + \psi(y^\alpha) \quad (\alpha \in \mathbb{R}) \]
holds for all \( x, y > 0 \) if and only if \( \alpha = \alpha_0 = -1.0266 \ldots \). Here, \( \alpha_0 \) is given by
\[ 2^{\alpha_0} = \inf_{t>0} \frac{\psi^{-1}(2\psi(t))}{t} = 0.4908 \ldots . \]

(ii) The inequality
\[ \psi(x^\beta) + \psi(y^\beta) \leq \psi((x+y)^\beta) \quad (\beta \in \mathbb{R}) \]
is valid for all \( x, y > 0 \) if and only if \( \beta = 0 \).

1. Introduction. Euler’s classical gamma function is defined for positive real numbers \( x \) by
\[ \Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt = e^{-\gamma x} \prod_{k=1}^{\infty} \left( 1 + \frac{x}{k} \right)^{-1} e^{x/k}. \]
We are concerned with the logarithmic derivative of the \( \Gamma \)-function,
\[ \psi(x) = \frac{d}{dx} \log \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \]
which is known as psi (or digamma) function. In view of its relevance in various fields, like, for example, the theory of special functions, statistics and mathematical physics, the \( \psi \)-function has been the subject of intensive work, and many interesting properties were discovered. Here are some of them:
Series and integral representations:

\[ \psi(x) = -\gamma - \frac{1}{x} + \sum_{k=1}^{\infty} \frac{x}{k(k+x)} = -\gamma + \int_{0}^{\infty} \frac{e^{-t} - e^{-xt}}{1-e^{-t}} \, dt. \]

Asymptotic formula:

\[ \psi(x) \sim \log(x) - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \cdots \quad (x \to \infty). \]

Reflection and recurrence formulas:

\[ \psi(1-x) = \psi(x) + \pi \cot(\pi x), \quad \psi(x+1) = \psi(x) + \frac{1}{x}. \]

Additional information on the \( \psi \)-function can be found, for instance, in \[1, \text{Chapter 6}\].

Numerous research articles were published in the recent past providing inequalities for the gamma and psi functions and their derivatives. We refer to the detailed bibliography \[23\] as well as to \[4, 14, 18, 24, 25, 26, 27, 28, 29\], and the references given therein.

In this paper, we are interested in certain sub- and super-additive properties of \( \psi \). We recall that a function \( f : (0, \infty) \to \mathbb{R} \) is said to be sub-additive, if

\[ f(x+y) \leq f(x) + f(y) \quad \text{for all} \quad x, y > 0. \]

If the converse inequality holds, then \( f \) is called super-additive. These functions have applications in different mathematical branches, like, for example, semi-group theory and functional analysis, and also in economics. See \[5, 6, 7, 10, 11, 12, 16\], \[17, \text{Chapter 16}\], \[19, 20, 22\] for more information on this subject.

We ask: do there exist real parameters \( \alpha \) and \( \beta \) such that \( \psi(x^\alpha) \) is subadditive and that \( \psi(x^\beta) \) is super-additive on \( (0, \infty) \)? It is our aim to answer this question. In the next section, we collect some lemmas which we need to prove our main results. In Section 3, we determine all \( \alpha, \beta \in \mathbb{R} \) such that the inequalities

\[ \psi((x+y)^\alpha) \leq \psi(x^\alpha) + \psi(y^\alpha) \]

and

\[ \psi(x^\beta) + \psi(y^\beta) \leq \psi((x+y)^\beta) \]
are valid for all $x, y > 0$. We conclude our paper with a few remarks. Among others, we study convexity and concavity properties of $\psi(x^\alpha)$. These remarks are given in Section 4.

The numerical values have been calculated via the computer program MAPLE V, Release 5.1.

2. Lemmas. Throughout this paper, we denote by $x_0 = 1.4616\ldots$ the only positive zero of $\psi$. The first three lemmas provide known monotonicity properties of functions which are defined in terms of the $\psi$-function. Proofs are given in [2, 3].

**Lemma 1.** The function
\[ x \mapsto x \psi(x) \]
is strictly decreasing on $(0, r_0]$, where $r_0 = 0.2160\ldots$.

**Lemma 2.** Let $k \in \mathbb{N}$. The function
\[ (2.1) \quad \tau_k(x) = x^{k+1} |\psi^{(k)}(x)| \]
is strictly increasing on $(0, \infty)$.

**Lemma 3.** Let $k \in \mathbb{N}$. The function
\[ (2.2) \quad \Delta_k(x) = x \frac{\psi^{(k+1)}(x)}{\psi^{(k)}(x)} \]
is strictly increasing on $(0, \infty)$.

**Lemma 4.** Let
\[ (2.3) \quad P_b(x) = x^b \psi'(x) \quad (b \in \mathbb{R}). \]

(i) If $b < 1.98$, then $P_b$ is strictly decreasing on $(0, 0.08)$.
(ii) If $1.97 < b < 1.98$, then $P_b$ is strictly convex on $(0, x_0)$.

**Proof.** (i) Let $b < 1.98$ and $0 < x < 0.08$. Differentiation leads to
\[ x^{-b} P'_b(x) = b \frac{\psi'(x)}{x} + \psi''(x). \]
Since $\psi'$ is positive on $(0, \infty)$, we obtain

\[(2.4) \quad x^{-b} P'_b(x) < 1.98 \frac{\psi'(x)}{x} + \psi''(x) = \frac{\psi'(x)}{x} [1.98 + \Delta_1(x)],\]

where $\Delta_1$ is defined in (2.2). Applying Lemma 3 gives

\[(2.5) \quad \Delta_1(x) < \Delta_1(0.08) = -1.982 \ldots \]

Combining (2.4) and (2.5) reveals that $P'_b(x) < 0$.

(ii) Let $1.97 < b < 1.98$. We consider two cases.

Case 1. $0 < x < 0.1$. By differentiation, we get

\[x^{4-b} P''_b(x) = (b^2 - b) \tau_1(x) - 2b \tau_2(x) + \tau_3(x),\]

where $\tau_1$, $\tau_2$, $\tau_3$ are defined in (2.1). Using $\tau_1(0) = \lim_{x \to 0} \tau_1(x) = 1$, $\tau_3(0) = \lim_{x \to 0} \tau_3(x) = 6$, and Lemma 2 yields

\[x^{4-b} P''_b(x) \geq b^2 - b - 2b \tau_2(0.1) + 6 = q(b), \text{ say.}\]

Since $q$ is decreasing on $[1.97, 1.98]$ with $q(1.98) = 0.013 \ldots$, we obtain $P''_b(x) > 0$.

Case 2. $0.1 \leq x \leq x_0$. We have

\[x^{2-b} \frac{P''_b(x)}{\psi'(x)} = b^2 - b + 2b \Delta_1(x) + \Delta_1(x) \Delta_2(x),\]

with $\Delta_1$ and $\Delta_2$ as defined in (2.2). Since $\Delta_1$ and $\Delta_2$ are negative and increasing, we conclude that the product $\Delta_1 \Delta_2$ is decreasing. Let $0.1 \leq r \leq x \leq s \leq x_0$. Then, we get

\[x^{2-b} \frac{P''_b(x)}{\psi'(x)} \geq 1.97^2 - 1.97 + 2 \cdot 1.98 \Delta_1(r) + \Delta_1(s) \Delta_2(s)\]

\[= K(r, s), \text{ say.}\]

Since

\[K \left(0.1 + \frac{k}{150}, 0.1 + \frac{k + 1}{150}\right) > 0\]

for $k = 0, 1, \ldots, 203$ and $K(1.46, x_0) = 0.037 \ldots$, we find that $P''_b$ is positive on $(0, x_0)$.

**Lemma 5.** Let $c = 0.49084$. For $t > 0$ we have

\[\psi(ct) < 2\psi(t).\]
Proof. To show that \( g(t) = 2\psi(t) - \psi(ct) \) is positive on \((0, \infty)\) we consider three cases.

Case 1. \( 0 < t \leq 0.2 \). Let \( 0 < a \leq 1.08 \) and \( h_a(t) = t\psi(at) \). Since \( 0 < at \leq 0.216 \), we conclude from Lemma 1 that \( h_a \) is decreasing on \((0, 0.2]\). We set \( 0 \leq r \leq t \leq s \leq 0.2 \) and obtain

\[
\begin{align*}
tg(t) &= 2h_1(t) - h_c(t) \\
&\geq 2h_1(s) - h_c(r) = H(r, s), \quad \text{say.}
\end{align*}
\]

By direct computation, we find

\[
H(0, 0.03) = \frac{1}{22} \
\]

\[
H\left(0.03 + \frac{k}{100}, 0.03 + \frac{k+1}{100}\right) > 0 \quad \text{for } k = 0, 1, 2, 3, 4,
\]

\[
H\left(0.08 + \frac{k}{400}, 0.08 + \frac{k+1}{400}\right) > 0 \quad \text{for } k = 0, 1, \ldots, 12,
\]

\[
H\left(0.1125 + \frac{k}{2000}, 0.1125 + \frac{k+1}{2000}\right) > 0 \quad \text{for } k = 0, 1, \ldots, 27,
\]

\[
H\left(0.1265 + \frac{k}{12000}, 0.1265 + \frac{k+1}{12000}\right) > 0 \quad \text{for } k = 0, 1, \ldots, 101,
\]

\[
H\left(0.135 + \frac{k}{2000}, 0.135 + \frac{k+1}{2000}\right) > 0 \quad \text{for } k = 0, 1, \ldots, 99,
\]

\[
H\left(0.14 + \frac{k}{8000}, 0.14 + \frac{k+1}{8000}\right) > 0 \quad \text{for } k = 0, 1, \ldots, 79,
\]

\[
H\left(0.15 + \frac{k}{900}, 0.15 + \frac{k+1}{900}\right) > 0 \quad \text{for } k = 0, 1, \ldots, 44.
\]

This leads to \( g(t) > 0 \) for \( t \in (0, 0.2] \).

Case 2. \( 0.2 < t \leq x_0 \). Let \( 0.2 \leq r \leq t \leq s \leq x_0 \). Since \( \psi \) is strictly increasing on \((0, \infty)\), we obtain

\[
g(t) \geq 2\psi(r) - \psi(cs) = I(r, s), \quad \text{say.}
\]

We have

\[
\begin{align*}
I\left(0.2 + \frac{k}{1500}, 0.2 + \frac{k+1}{1500}\right) > 0 & \quad \text{for } k = 0, 1, \ldots, 149, \\
I\left(0.3 + \frac{k}{160}, 0.3 + \frac{k+1}{160}\right) > 0 & \quad \text{for } k = 0, 1, \ldots, 15,
\end{align*}
\]
\[ I\left(0.4 + \frac{k}{50}, 0.4 + \frac{k+1}{50}\right) > 0 \quad \text{for} \quad k = 0, 1, \ldots, 29, \]
\[ I(1, x_0) = 0.017 \ldots. \]

Thus, \( g(t) > 0 \) for \( t \in [0.2, x_0] \).

**Case 3.** \( t \geq x_0 \). Since \( \psi(t) \geq 0 \) and \( \psi(t) > \psi(ct) \), we get \( g(t) = \psi(t) + [\psi(t) - \psi(ct)] > 0 \). \( \Box \)

**Lemma 6.** Let \( \alpha \in [-1.027, -1.026] \) and 
\[ F_\alpha(s, t) = \psi(s) + \psi(t) - \psi([s^{1/\alpha} + t^{1/\alpha}]^\alpha). \]
There exists a function \( \sigma_\alpha \) such that \( F_\alpha(s, t) \geq \sigma_\alpha(s) \) for all \( s, t \in \mathbb{R} \) with \( 0 < s \leq t \leq x_0 \) and \( \lim_{s \to 0} \sigma_\alpha(s) = \infty \).

**Proof.** We distinguish two cases.

**Case 1.** \( t < 0.08 \). We have \( 0 < s \leq t < 0.08 \). Partial differentiation yields 
\[ t^{1-1/\alpha} \frac{\partial}{\partial t} F_\alpha(s, t) = P_a(t) - P_a([s^{1/\alpha} + t^{1/\alpha}]^\alpha) \]
with \( a = 1 - 1/\alpha \) and \( P_b \) as defined in (2.3). Since 
\[ a < 1.98 \quad \text{and} \quad 0 < [s^{1/\alpha} + t^{1/\alpha}]^\alpha < t < 0.08, \]
we conclude from Lemma 4 (i) that \( (\partial/\partial t)F_\alpha(s, t) < 0 \). This implies that \( t \mapsto F_\alpha(s, t) \) is strictly decreasing on \([s, 0.08] \). Thus, we obtain
\[ (2.6) \quad F_\alpha(s, t) \geq F_\alpha(s, 0.08). \]

Let \( A = [s^{1/\alpha} + 0.08]^{1/\alpha} \). We have \( 0 < A < 0.08 \). Using the identity \( \psi(x) = \psi(x + 1) - 1/x \) and the monotonicity of \( \psi \) gives 
\[ (2.7) \quad F_\alpha(s, 0.08) = \psi(0.08) + \psi(s + 1) - \psi(A + 1) + \frac{1}{A} - \frac{1}{s} \geq \psi(0.08) + \psi(1) - \psi(1.08) + \frac{1}{A} - \frac{1}{s}. \]

Since \( -\alpha > 1 \), we get 
\[ (2.8) \quad \frac{1}{A} - \frac{1}{s} = \frac{1}{s} \left[ \left(1 + \left(\frac{0.08}{s}\right)^{1/\alpha}\right)^{-\alpha} - 1 \right] \geq 0.08^{1/\alpha} s^{-1-1/\alpha}. \]
Combining (2.6)–(2.8) gives

\[ F_\alpha(s, t) \geq c_0 + 0.08^{1/\alpha} s^{-1-1/\alpha} \]

with \( c_0 = \psi(0.08) + \psi(1) - \psi(1.08) = -13.077 \ldots \).

**Case 2.** \( 0.08 \leq t \). We have \( 0 < s \leq t \leq x_0 \) and \( 0.08 \leq t \). Let \( B = [s^{1/\alpha} + t^{1/\alpha}]^\alpha \). Then, \( B \leq 2^{-1.026} x_0 \). We obtain

\[ F_\alpha(s, t) \geq \psi(s) + \psi(0.08) - \psi(B) \]

\[ = \psi(0.08) + \psi(s + 1) - \psi(B + 1) + \frac{1}{B} - \frac{1}{s} \]

\[ \geq \psi(0.08) + \psi(1) - \psi(2^{-1.026} x_0 + 1) + \frac{1}{B} - \frac{1}{s}. \]

Furthermore,

\[ \frac{1}{B} - \frac{1}{s} = \frac{1}{s} \left[ \left( 1 + \left( \frac{t}{s} \right)^{1/\alpha} \right)^{-\alpha} - 1 \right] \]

\[ \geq \frac{1}{s} \left( \frac{t}{s} \right)^{1/\alpha} \]

\[ \geq x_0 \alpha s^{-1-1/\alpha}. \]

From (2.10) and (2.11), we get

\[ F_\alpha(s, t) \geq c_0^* + x_0^{1/\alpha} s^{-1-1/\alpha} \]

with \( c_0^* = \psi(0.08) + \psi(1) - \psi(2^{-1.026} x_0 + 1) = -13.752 \ldots \).

We have \( x_0^{1/\alpha} < 0.08^{1/\alpha} \) and \( c_0^* < c_0 \). Thus, from (2.9) and (2.12), we obtain for \( s, t \in \mathbb{R} \) with \( 0 < s \leq t \leq x_0 \):

\[ F_\alpha(s, t) \geq c_0^* + x_0^{1/\alpha} s^{-1-1/\alpha} = \sigma_\alpha(s), \quad \text{say.} \]

Since \(-1 - 1/\alpha < 0\), we conclude that \( \lim_{s \to 0} \sigma_\alpha(s) = \infty \).

### 3. Main results.

We are now ready to describe completely the sub- and super-additive properties of \( \psi(x^\alpha) \).

**Theorem 1.** Let \( \alpha \) be a real number. The inequality

\[ \psi((x + y)^\alpha) \leq \psi(x^\alpha) + \psi(y^\alpha) \]

holds for all positive real numbers \( x \) and \( y \) if and only if

\[ \alpha \leq \alpha_0 = -1.0266 \ldots \]
Here, \( \alpha_0 \) is given by

\[
(3.3) \quad 2^{\alpha_0} = \inf_{t > 0} \frac{\psi^{-1}(2\psi(t))}{t} = 0.4908\ldots.
\]

(As usual, \( \psi^{-1} \) denotes the inverse function of \( \psi \).)

**Proof.** First, we assume that (3.1) is valid for all \( x, y > 0 \). If \( \alpha > 0 \), then we obtain

\[
\lim_{x \to 0} \psi((x + y)^\alpha) = \psi(y^\alpha) \quad \text{and} \quad \lim_{x \to 0} [\psi(x^\alpha) + \psi(y^\alpha)] = -\infty,
\]
a contradiction. Hence, \( \alpha \leq 0 \). If \( \alpha = 0 \), then (3.1) is equivalent to \( 0 \leq \psi(1) \). But, \( \psi(1) = -\gamma < 0 \). It follows that \( \alpha < 0 \). We set \( x = y \) and \( t = x^\alpha \). Then, we get for \( t > 0 \):

\[
\psi(2^\alpha t) \leq 2\psi(t).
\]

Therefore,

\[
2^\alpha \leq \frac{\psi^{-1}(2\psi(t))}{t} = J(t), \quad \text{say}.
\]

This gives \( \alpha \leq \alpha_0 \), where \( 2^{\alpha_0} = \inf_{t > 0} J(t) \). From Lemma 5 we conclude that

\[
0.49084 < J(t) \quad \text{for} \quad t > 0,
\]

so that \( J(0.13654) = 0.49084\ldots \) leads to

\[
0.49084 \leq 2^{\alpha_0} \leq 0.49084\ldots.
\]

It follows that

\[
2^{\alpha_0} = 0.4908\ldots \quad \text{and} \quad \alpha_0 = -1.0266\ldots.
\]

Next, we prove: if \( \alpha \leq \alpha_0 \) with \( \alpha_0 \) as given in (3.2) and (3.3), respectively, then (3.1) is valid for all \( x, y > 0 \). We set \( x = s^{1/\alpha} \) and \( y = t^{1/\alpha} \). Then, (3.1) can be written as

\[
\psi([s^{1/\alpha} + t^{1/\alpha}]^\alpha) \leq \psi(s) + \psi(t).
\]

Since \( \psi \) is increasing on \((0, \infty)\) and \( a \mapsto [s^{1/a} + t^{1/a}]^a \) is increasing on \((-\infty, 0)\), see [8, page 18], we obtain

\[
\psi([s^{1/\alpha} + t^{1/\alpha}]^\alpha) \leq \psi([s^{1/\alpha_0} + t^{1/\alpha_0}]^{\alpha_0}).
\]
Hence, it suffices to show that, if \( 0 < s \leq t \), then
\[
\psi \left( [s^{1/\alpha_0} + t^{1/\alpha_0}]^{\alpha_0} \right) \leq \psi(s) + \psi(t).
\]

To prove (3.4) we consider two cases.

**Case 1.** \( x_0 < t \). Since \([s^{1/\alpha_0} + t^{1/\alpha_0}]^{\alpha_0} < s\), we get
\[
\psi \left( [s^{1/\alpha_0} + t^{1/\alpha_0}]^{\alpha_0} \right) < \psi(s) < \psi(s) + \psi(t).
\]

**Case 2.** \( t \leq x_0 \). Let \( W = \{(s, t) \in \mathbb{R}^2 \mid 0 < s < t \leq x_0\} \) and
\[
F(s, t) = \psi(s) + \psi(t) - \psi \left( [s^{1/\alpha_0} + t^{1/\alpha_0}]^{\alpha_0} \right).
\]

We set
\[
M = \max_{0.01 \leq s \leq t \leq x_0} F(s, t).
\]

Applying Lemma 6 (with \( \alpha = \alpha_0 \)) reveals that there exists a number \( \delta > 0 \) such that, for all \( s, t \in \mathbb{R} \) with \( 0 < s < \delta \) and \( s \leq t \leq x_0 \), we have \( F(s, t) \geq M \). Let \( \delta^* = \min\{\delta, 0.01\} \). We show that, for all \((\tilde{s}, \tilde{t}) \in W\) we have
\[
F(\tilde{s}, \tilde{t}) \geq \min_{\delta^* \leq s \leq t \leq x_0} F(s, t).
\]

**Case 2.1.** \( \delta^* \leq \tilde{s} \). Then we have \( \delta^* \leq \tilde{s} \leq \tilde{t} \leq x_0 \). This implies that (3.5) holds.

**Case 2.2.** \( \tilde{s} \leq \delta^* \). Then, \( 0 < \tilde{s} \leq \delta \), \( \tilde{s} \leq \tilde{t} \leq x_0 \) and \( \delta^* \leq 0.01 \). It follows that
\[
F(\tilde{s}, \tilde{t}) \geq M \geq \min_{0.01 \leq s \leq t \leq x_0} F(s, t) \geq \min_{\delta^* \leq s \leq t \leq x_0} F(s, t).
\]

Thus, there exist real numbers \( s_0, t_0 \) with \((s_0, t_0) \in W\) such that \( F(s, t) \geq F(s_0, t_0) \) for all \((s, t) \in W\). We suppose that \((s_0, t_0)\) is an interior point of \( W \). Then we obtain
\[
s_0 \frac{\partial F(s, t)}{\partial s} \bigg|_{(s, t) = (s_0, t_0)} = P_{b_0} (s_0) - P_{b_0} (C) = 0
\]
and
\[
t_0 \frac{\partial F(s, t)}{\partial t} \bigg|_{(s, t) = (s_0, t_0)} = P_{b_0} (t_0) - P_{b_0} (C) = 0
\]
where \( b_0 = 1 - 1/\alpha_0 \), \( C = [s_0^{1/\alpha_0} + t_0^{1/\alpha_0}]^{\alpha_0} \), and \( P_b \) as defined in (2.3). It follows that
\[
P_{b_0}(s_0) = P_{b_0}(t_0) = P_{b_0}(C)
\]
with \( 0 < C < s_0 < t_0 < x_0 \) and \( 1.97 < b_0 < 1.98 \). This contradicts Lemma 4 (ii). Thus, we have either \( 0 < s_0 = t_0 \leq x_0 \) or \( 0 < s_0 \leq t_0 = x_0 \). In the first case, we obtain
\[
F(s_0, t_0) = 2\psi(t_0) - \psi(2^{\alpha_0}t_0).
\]
Since
\[
2^{\alpha_0} \leq \frac{\psi^{-1}(2\psi(t_0))}{t_0},
\]
we get \( F(s_0, t_0) \geq 0 \). And, the second case leads to
\[
F(s_0, t_0) = \psi(s_0) - \psi(C) > 0.
\]
The proof of Theorem 1 is complete.

\[\Box\]

**Theorem 2.** Let \( \beta \) be a real number. The inequality
\[
\psi(x^\beta) + \psi(y^\beta) \leq \psi((x+y)^\beta)
\]
is valid for all positive real numbers \( x \) and \( y \) if and only if \( \beta = 0 \).

**Proof.** Since \( \psi(1) = -\gamma \), we conclude that (3.6) holds if \( \beta = 0 \). Next, we assume that (3.6) is valid for all \( x, y > 0 \). If \( \beta < 0 \), then the sum on the left-hand side tends to \( \infty \) as \( x \to 0 \), whereas the right-hand side converges to \( \psi(y^\beta) \). Hence, \( \beta \geq 0 \). We suppose that \( \beta > 0 \) and set \( x = y, t = x^\beta \). Then, (3.6) reads
\[
2\psi(t) \leq \psi(2^\beta t).
\]
This yields for \( t > 1 \):
\[
2 \frac{\psi(t)}{\log(t)} \leq \frac{\psi(2^\beta t)}{\log(2^\beta t)} \left(1 + \frac{\beta \log(2)}{\log(t)}\right).
\]
Applying \( \lim_{t \to \infty} \psi(t)/\log(t) = 1 \) leads to \( 2 \leq 1 \). This contradiction gives \( \beta = 0 \). \[\Box\]
4. Final remarks. (I) In what follows, we set $\Phi(x) = \psi(x^\alpha)$. The inequalities (3.1) and (3.6) are related to Jensen’s inequality and its converse. Therefore, it is natural to ask for all real parameters $\alpha$ such that $\Phi$ is convex/concave on $(0, \infty)$.

**Remark 1.** The inequality

$$
\psi\left(\left(\frac{x+y}{2}\right)^\alpha\right) < \frac{\psi(x^\alpha) + \psi(y^\alpha)}{2} \quad (\alpha \in \mathbb{R} \setminus \{0\})
$$

holds for all $x, y > 0$ with $x \neq y$ if and only if $\alpha \in [-1, 0)$. The converse of (4.1) is valid for all $x, y > 0$ with $x \neq y$ if and only if $\alpha > 0$.

**Proof.** Differentiation gives for $\alpha \neq 0$:

$$
\frac{x^{2-\alpha}}{\alpha^2 \psi'(x^\alpha)} \Phi''\alpha'(x) = \Delta_1(x^\alpha) + 1 - \frac{1}{\alpha},
$$

where $\Delta_1$ is defined in (2.2). Using this identity as well as Lemma 3 (with $k = 1$) and the limit relations

$$
\lim_{t \to 0} \Delta_1(t) = -2, \quad \lim_{t \to \infty} \Delta_1(t) = -1,
$$

we conclude that $\Phi''\alpha'(x) > 0$ for $x > 0$ if and only if $-1 \leq \alpha < 0$, and $\Phi''\alpha'(x) < 0$ for $x > 0$ if and only if $\alpha > 0$. \hfill \Box

(II) An application of Remark 1 leads to the following functional inequality.

**Remark 2.** The inequality

$$
\psi((x+y)^\alpha) + \psi(z^\alpha) \leq \psi(x^\alpha) + \psi((y+z)^\alpha) \quad (\alpha \in \mathbb{R} \setminus \{0\})
$$

holds for all $x, y, z > 0$ with $x \leq z$ if and only if $\alpha \in [-1, 0)$.

**Proof.** Let $-1 \leq \alpha < 0$ and

$$
Q_\alpha(x, y, z) = \Phi_\alpha(x) + \Phi_\alpha(y+z) - \Phi_\alpha(x+y) - \Phi_\alpha(z).
$$

Since $\Phi_\alpha$ is convex on $(0, \infty)$, we obtain

$$
\frac{\partial}{\partial y} Q_\alpha(x, y, z) = \Phi'_\alpha(y+z) - \Phi'_\alpha(x+y) \geq 0.
$$

This gives

$$
Q_\alpha(x, y, z) \geq Q_\alpha(x, 0, z) = 0.
$$
Let \( Q(\alpha(x,y,z)) \geq 0 \) for all \( x, y, z > 0 \) with \( x \leq z \). If \( \alpha > 0 \), then \( \lim_{x \to 0} Q(\alpha(x,y,z)) = -\infty \). This contradiction leads to \( \alpha < 0 \). Then, for \( z \geq x \), we get

\[
Q(\alpha(x,x,z)) = \Phi(\alpha(x) + \Phi(\alpha(x) + z) - \Phi(\alpha(2x) - \Phi(\alpha(z)) \geq 0
\]

This gives

\[
(4.2) \ 0 \leq (2x)^{\alpha+1} \frac{d}{dz} Q(\alpha(x,x,z)) \bigg|_{z=x} = \alpha \left[ (2x)^{2\alpha} \psi((2x)^\alpha) - 2^{\alpha+1}x^{2\alpha} \psi'(x^\alpha) \right].
\]

We let \( x \) tend to \( \infty \) and make use of the limit relation \( \lim_{t \to 0} t^2 \psi'(t) = 1 \). Then, (4.2) leads to \( 0 \leq \alpha(1 - 2^{\alpha+1}) \). Thus, \( \alpha \geq -1 \). \( \square \)

(III) The weighted power mean of order \( r \) is defined for positive real numbers \( a_1, \ldots, a_n \) and \( w_1, \ldots, w_n \) with \( w_1 + \cdots + w_n = 1 \) by

\[
M(r) = \left( \sum_{k=1}^{n} w_k a_k^r \right)^{1/r} \quad (r \in \mathbb{R} \setminus \{0\}).
\]

The main properties of this family of mean-values are collected in [15, Chapter 2]. In 1972, Beesack [9] presented a proof for the following remarkable inequality:

\[
(4.3) \ \frac{M(t) - M(r)}{M(t) - M(s)} < \frac{s(t-r)}{r(t-s)} \quad (0 < r < s < t).
\]

The validity of (4.3) for the special case \( w_1 = \cdots = w_n = 1/n \) was conjectured by Hsu in 1955. Here is a counterpart of (4.3) for the psi function.

**Remark 3.** The inequality

\[
(4.4) \ \frac{\psi(t^\alpha) - \psi(r^\alpha)}{\psi(t^\alpha) - \psi(s^\alpha)} < \frac{s(t-r)}{r(t-s)} \quad (\alpha \in \mathbb{R} \setminus \{0\})
\]

holds for all real numbers \( r, s, t \) with \( 0 < r < s < t \) if and only if \( \alpha < 0 \) or \( 0 < \alpha \leq 1 \).

**Proof.** Let \( 0 < r < s < t \). To prove (4.4) we consider two cases.
Case 1. \(0 < \alpha \leq 1\). Since \(\Phi_{-\alpha}\) is strictly convex on \((0, \infty)\), we obtain for \(x, y > 0\) with \(x \neq y\) and \(\lambda \in (0, 1)\):

\[
\Phi_{-\alpha}(\lambda x + (1 - \lambda)y) < \lambda \Phi_{-\alpha}(x) + (1 - \lambda)\Phi_{-\alpha}(y).
\]

We set

\[
x = \frac{1}{t}, \quad y = \frac{1}{r}, \quad \text{and} \quad \lambda = \frac{t(s-r)}{s(t-r)}.
\]

Then, (4.5) gives

\[
\Phi_{\alpha}(s) < \frac{t(s-r)}{s(t-r)} \Phi_{\alpha}(t) + \frac{r(t-s)}{s(t-r)} \Phi_{\alpha}(r).
\]

This is equivalent to

\[
(4.6) \quad r(t-s)[\Phi_{\alpha}(t) - \Phi_{\alpha}(s)] < s(t-r)[\Phi_{\alpha}(t) - \Phi_{\alpha}(s)].
\]

The function \(\Phi_{\alpha}\) is strictly increasing on \((0, \infty)\), so that (4.6) implies (4.4).

Case 2. \(\alpha < 0\). The strict concavity of \(\Phi_{-\alpha}\) reveals that (4.5) and (4.6) are valid with “>” instead of “<.” Since \(\Phi_{\alpha}\) is strictly decreasing on \((0, \infty)\), we conclude that (4.6) leads to (4.4).

Conversely, let (4.4) be valid for all \(r, s, t\) with \(0 < r < s < t\). We assume that \(\alpha > 1\). Then we get

\[
R_{\alpha}(r, t) < R_{\alpha}(s, t)
\]

with

\[
R_{\alpha}(x, t) = x \frac{\psi(t^{\alpha}) - \psi(x^{\alpha})}{t - x}.
\]

Let \(0 < x < t\). We obtain

\[
(4.7) \quad 0 \leq (t-x)^2 x^{\alpha} \frac{\partial}{\partial x} R_{\alpha}(x, t)
= -t[x^{\alpha}\psi(x^{\alpha}) - x^{\alpha}\psi(t^{\alpha})] + \alpha(x-t)x^{2\alpha}\psi'(x^{\alpha}),
\]

and let \(x\) tend to 0. Then, the expression on the right-hand side of (4.7) converges to \(t(1 - \alpha)\). Hence, \(\alpha \leq 1\).

(IV) The logarithmic mean of two positive real numbers \(x, y\) with \(x \neq y\) is defined by

\[
L(x, y) = \frac{x - y}{\log(x) - \log(y)}.
\]
This mean value plays a role not only in mathematics, but it also has applications in physics and economics. For more information on this subject we refer to [21] and the references given therein. In 2008, Chu et al. [13] proved an elegant inequality involving the psi function and the logarithmic mean:

\[(4.8) \quad (y - x)\psi(\sqrt{xy}) < (L(x, y) - x)\psi(x) + (y - L(x, y))\psi(y) \quad (2 \leq x < y).\]

The authors conjectured that (4.8) is valid for all \(x, y > 0\) with \(y > x\). However, this conjecture is not true. To show this we set \(y = x_0\) and multiply both sides of (4.8) by \(\sqrt{x_0x}\). This leads to

\[
(x_0 - x)\sqrt{x_0x}\psi(\sqrt{x_0x}) < (L(x, x_0) - x)\sqrt{x_0x}\psi(x) = (-x\psi(x)) \left[ \sqrt{x_0x} + \frac{x_0 - x}{2\sqrt{x/x_0}\log\sqrt{x/x_0}} \right].
\]

If \(x\) tends to 0, then the expression on the left-hand side converges to \(-x_0\), whereas the right-hand side tends to \(-\infty\), a contradiction.

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REFERENCES


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