ON A LOGARITHMIC HARDY-BLOCH TYPE SPACE

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ABSTRACT. In this paper, given $0 < p < \infty$, we define a logarithmic Hardy-Bloch type space

$$BH_{p,L} = \{ f(z) \in H(D) : \| f \|_{p,L} = \sup_{z \in D} (1 - |z|) \log \frac{e}{1 - |z|} M_p(|z|, f') < \infty \}.$$ 

Then we mainly study the relation between $BH_{p,L}$ and three classical spaces: Hardy space, Dirichlet type space and VMOA. We also obtain some estimates on the growth of $f \in BH_{p,L}$.

1. Introduction. Let $D = \{ z : |z| < 1 \}$ be the open unit disk in the complex plane $C$, and let $H(D)$ denote the set of all analytic functions on $D$. For $0 \leq r < 1$, $f(z) \in H(D)$, we set

$$M_p(r, f) = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{1/p}, \quad 0 < p < \infty,$$

$$M_{\infty}(r, f) = \max_{0 \leq \theta \leq 2\pi} |f(re^{i\theta})|.$$ 

For $0 < p \leq \infty$, the Hardy space $H^p$ consists of those functions $f \in H(D)$, for which

$$\| f \|_{H^p} = \sup_{0 \leq r < 1} M_p(r, f) < \infty.$$ 

The Dirichlet type space $D^p_{p-1}$ consists of those functions $f \in H(D)$,

2010 AMS Mathematics subject classification. Primary 30H10, 30H30, 30H35.
Keywords and phrases. Hardy space, Bloch space, Dirichlet type space.

The research was supported by National Natural Science Foundation of China (grant No. 11271124) and Natural Science Foundation of Fujian Province, China (grant No. 2009J01004).
The second author is the corresponding author.
Received by the editors on January 4, 2010, and in revised form on June 18, 2012.
DOI:10.1216/RMJ-2014-44-5-1669 Copyright ©2014 Rocky Mountain Mathematics Consortium

1669
for which

\[ \int_D (1 - |z|)^{p-1} |f'(z)|^p dA(z) < \infty, \]

where \( dA(z) = 1/\pi \, dx \, dy \) denotes the norm Lebesgue area measure on \( D \). Hence, \( f \in \mathcal{D}_p \) if and only if

\[ (1) \quad \int_D (1 - |z|)^{p-1} |f'(z)|^p dA(z) = 2\pi \int_0^1 r(1-r)^{p-1} M_p(r, f') \, dr < \infty. \]

The \( \mathcal{D}_p \) is closely related to \( H^p \). A classical result of Littlewood and Paley [9] (see also [10]) asserts that

\[ H^p \subset \mathcal{D}_p, \quad 2 \leq p < \infty. \]

On the other hand, see, e.g., [11, 14], we have

\[ \mathcal{D}_p \subset H^p, \quad 0 < p \leq 2. \]

These inclusions are strict if \( p \neq 2 \).

Given \( 0 < p \leq \infty \) and \( 0 \leq \alpha < \infty \), we write \( BH_{p,\alpha} \) and \( BH_{p,L} \) for the spaces of those \( f(z) \in H(D) \), such that

\[ BH_{p,\alpha} = \left\{ f(z) \in H(D) : ||f||_{p,\alpha} = \sup_{z \in D} (1 - |z|)^\alpha M_p(|z|, f') < \infty \right\}, \]

\[ BH_{p,L} = \left\{ f(z) \in H(D) : ||f||_{p,L} = \sup_{z \in D} (1 - |z|) \log \frac{e}{1 - |z|} M_p(|z|, f') < \infty \right\}. \]

It is easy to prove that \( BH_{p,\alpha} \) and \( BH_{p,L} \) are complete under the norms

\[ ||f||_{p,\alpha} = ||f||_{p,\alpha} + |f(0)|, \]

\[ ||f||_{p,L} = ||f||_{p,L} + |f(0)|. \]

When \( p \geq 1 \), the two spaces above are Banach spaces.

For the two spaces above, it is evident that \( BH_{p,L} \supseteq BH_{q,L} \) for \( 0 < p < q \leq \infty \). With the terminology just introduced, we have \( BH_{\infty,\alpha} = \mathcal{B}_\alpha \) and \( BH_{\infty,L} = \mathcal{B}_L \), where \( \mathcal{B}_\alpha \) and \( \mathcal{B}_L \) denote the \( \alpha \)-Bloch space and the logarithmic Bloch space, respectively, the
properties on these two spaces are abundant. More information about \( \mathcal{B}_\alpha \) and \( \mathcal{B}_L \) can be found in [12, 15, 16, 17, 18, 20, 21].

When \( 0 \leq \alpha < 1 \), the following result is due to Hardy and Littlewood, see [8].

**Theorem A.** Let \( 0 \leq \alpha < 1 \) and \( 1 \leq p \leq \infty \). Then

\[
\mathcal{B}_{H, p} = \Lambda_{1-\alpha}^p = \{ f \in H^p : \omega_p(t, f) = O(t^{1-\alpha}), \text{ as } t \to 0 \},
\]

where

\[
\omega_p(t, f) = \sup_{0 < |h| \leq t} \left( \frac{1}{2\pi} \int_0^{2\pi} \left| f(e^{i(\theta + h)}) - f(e^{i\theta}) \right|^p \, d\theta \right)^{1/p}, \quad t > 0, \text{ if } 1 \leq p < \infty,
\]

\[
\omega_p(t, f) = \sup_{0 < |h| \leq t} \left( \text{ess sup} \left| f(e^{i(\theta + h)}) - f(e^{i\theta}) \right| \right), \quad t > 0.
\]

Thus, \( \mathcal{B}_{H, p, \alpha} \) is a mean Lipschitz space. On this basis, Blasco [2] and Girela and Mázquez [5] extend the result.

When \( \alpha = 1 \), the following result is known (see [7] and [6]).

**Theorem B.** Let \( f \in H(D) \), if \( 0 < p < \infty \) and \( f \in \mathcal{B}_{H, p, 1} \). Then

\[
M_p(r, f) = O\left( \left( \log \frac{1}{1-r} \right)^{\beta} \right),
\]

where

(i) \( \beta = 1/p \), for \( 0 < p < 2 \),

(ii) \( \beta = 1/2 \) for \( 2 \leq p < \infty \).

Our main goal in this paper is to show that, if \( f \in \mathcal{B}_{H, p, L} \), whose rate of growth \( M_p(r, f') \) (\( 1 < p < \infty \)) is between the ones for the functions in \( \mathcal{B}_{H, p, \alpha} \) (\( 0 < \alpha < 1 \)) and \( \mathcal{B}_{H, p, 1} \), then \( f \in (H^p \cap \mathcal{D}_{p-1}^p) \), see Theorem 2.1. We also characterize that \( \mathcal{B}_{H, p, L} \subset (H^p \cap \mathcal{D}_{p-1}^p) \) for \( p \in (1, \infty) \) is strict. However, the containment is not true for \( 0 < p \leq 1 \). We also note that \( \mathcal{B}_{H, p, L} \not\subset VMOA \) for every \( 0 < p < \infty \). As for the growth of \( f \in \mathcal{B}_{H, p, L} \), we give a sharp estimate.
Throughout this paper, $C$, which may change from one occurrence to the next, denotes a positive and finite constant only dependent on $p$ and $\alpha$.

2. Proof of main results.

**Theorem 2.1.** Suppose $1 < p < \infty$. Then $BH_{p,L} \subset (H^p \cap \mathcal{D}^p_{p-1})$.

To complete the proof, we need the following three lemmas.

**Lemma 2.1.** [6] If $2 < p < \infty$, then there is a constant $C_p$ depending only on $p$ such that

$$
\|f\|_{H^p} \leq C_p \left( |f(0)| + \left( \int_0^1 (1-r)M^2_p(r, f') \, dr \right)^{1/2} \right),
$$

for all $f \in H(D)$.

**Lemma 2.2.** [7] If $0 < p \leq 2$, then there is a constant $C_p$ depending only on $p$ such that

$$
\|f\|^p_{H^p} \leq C_p \left( |f(0)|^p + \int_D (1-|z|)^{p-1}|f'(z)|^p \, dA(z) \right),
$$

for every $f \in \mathcal{D}^p_{p-1}$.

**Lemma 2.3.** If $0 < \alpha, \beta < \infty$, $x \in (0,\epsilon]$, then $f(x) = x^\alpha (\log e/x)^\beta$ increases on $(0,\epsilon^{1-\beta/\alpha}]$, decreases on $[\epsilon^{1-\beta/\alpha},\epsilon]$.

The proof of this lemma is easy; we omit the details here.

*Proof of Theorem 2.1.* Take $f \in BH_{p,L}$ and assume, without loss of generality, that $f(0) = 0$. When $2 < p < \infty$, for $0 < r < 1$, set $f_r(z) = f(rz)$. Applying Lemma 2.1 to $f_r$, we obtain that

$$
M^2_p(r, f) \leq C \int_0^1 r^2(1-s)M^2_p(rs, f') \, ds
$$

$$
\leq C \int_0^1 \frac{1-s}{(1-rs)^2((\log e/(1-rs))^2} \, ds.
$$
Since $rs < s$, Lemma 2.3 implies
\[ M_p^2(r, f) \leq C \int_0^1 \frac{1}{(1-s)(\log e/(1-s))^2} ds < \infty. \]

When $1 < p \leq 2$, using Lemma 2.2 yields
\[
M_p^p(r, f) \leq C \int_D r^p (1 - |w|)^{p-1} |f'(rw)|^p dA(w) \\
\leq C \int_0^1 \frac{(1-s)^{p-1}}{(1-rs)^p (\log e/(1-rs))^p} ds \\
\leq C \int_0^1 \frac{1}{(1-s)(\log e/(1-s))^p} ds.
\]
Hence $f \in H^p$. The assertion that $f \in D_{p-1}^p$ can be easily obtained from (1) finishes the proof.

This theorem is not true for $0 < p \leq 1$ and $p = \infty$. Indeed, when $p = \infty$, the space $BH_{\infty,L} = \mathcal{B}_L$. We take $f(z) = \log \log e/(1-z)$. Then
\[
M_\infty(r, f') = O\left(\frac{1}{(1-r) \log e/(1-r)}\right),
\]
that is, $f \in \mathcal{B}_L$, but $f \notin H^\infty$. In the case $0 < p \leq 1$, we only prove the following result.

**Theorem 2.2.** Suppose $0 < p \leq 1$. Let
\[
f(z) = \frac{1}{(1-z)^{1/p} \log e/(1-z)}, \quad z \in D.
\]
Then $f \in BH_{p,L}$, but $f \notin H^p$.

The following three lemmas are needed in the proof of Theorem 2.2.

**Lemma 2.4.** If $a > 1$ and $0 < r < 1$, then there exist two constants
\[
C_1 = \frac{2}{a-1} \left(1 - \frac{1}{(1+\pi)^{a-1}}\right)
\]
and
\[
C_2 = \frac{4\pi^a}{a-1}.
\]
such that
\[ C_1 (1 - r)^{1-a} \leq \int_0^{2\pi} |re^{i\theta} - r|^{-a} \, dt \leq C_2 (1 - r)^{1-a}, \quad \rho = \frac{1}{2}(1 + r). \]

The proof is similar to [3, Lemma in Chapter 4.6], so we omit the details.

**Lemma 2.5.** For \(0 < \alpha, \beta < \infty\) and \(z \in D\), let
\[
f(z) = \frac{(1 - |z|)^\alpha (\log e/(1 - |z|))^\beta}{(|1 - z|)^\alpha (\log e/|1 - z|)^\beta}.
\]
Then \(f(z) \leq 1\) in the case \(\beta \leq \alpha \log e/2\), and \(f(z) \leq (e^{\alpha-\beta}(6\beta)\beta)/2^\alpha \alpha^\beta\) in the case \(\beta > \alpha \log e/2\).

**Proof.** For \(\beta \leq \alpha \log e/2\), by Lemma 2.3, we have
\[
(1 - |z|)^\alpha \left( \log \frac{e}{1 - |z|} \right)^\beta \leq (|1 - z|)^\alpha \left( \log \frac{e}{|1 - z|} \right)^\beta,
\]
for all \(z \in D\), which implies \(f(z) \leq 1\).

When \(\beta > \alpha \log e/2\), let \(z \in D_1 = \{z \in D, |1 - z| < e^{1-(\beta/\alpha)}\}\), we deduce that \(|f(z)| \leq 1\). On the other hand, the condition \(z \in D \setminus D_1\) implies that
\[
f(z) \leq \frac{e^{\alpha-\beta}(\beta/\alpha)\beta}{2^\alpha (\log e/2)^\beta} \leq \frac{6\beta e^{\alpha-\beta}(\beta/\alpha)\beta}{2^\alpha} \leq \frac{e^{\alpha-\beta}(6\beta)\beta}{2^\alpha \alpha^\beta}.
\]
This finishes the proof. \(\square\)

The following lemma may be found in [8].

**Lemma 2.6.** Suppose that \(0 < p < \infty\) and \(f \in H^p\). Then
\[
\int_0^1 M_\infty^p (r, f) \, dr \leq \pi \|f\|_{H^p}^p.
\]
Proof of Theorem 2.2. Take the function
\[ g_c(z) = \frac{z^c}{(1 - z) (\log 1/(1 - z))^c}. \]

Exercise 3 [3, Chapter 1] says that \( g_c \in H^1 \) \((c > 1)\). However, set \( z = re^{i\theta} \),
\[ \int_0^1 M_1^1(r, g_c) \, dr \geq \int_0^1 \frac{r^c}{(1 - r) (\log 1/(1 - r))^c} \, dr = \infty, \]
for every \( 0 \leq c \leq 1 \),
which, by Lemma 2.6, implies \( g_c(z) \notin H^1 \) for \( 0 \leq c \leq 1 \).

For \( r = |z| \geq 1/2 \), there exists a constant \( C > 0 \), such that \( |\log 1/(1 - z)| \geq C \). Therefore,
\[ \frac{|f(z)|^p}{|g_1(z)|} \geq 2 \frac{|\log 1/(1 - z)|}{|(\log e/(1 - z))^p|} \geq 2 \frac{|\log 1/(1 - z)| |\log e/(1 - z)|^{1-p}}{1 + |\log 1/(1 - z)|} \geq \frac{2C}{1 + C} \left( \frac{\log e}{2} \right)^{1-p}. \]

It follows that \( f \notin H^p \). On the other hand,
\[ f'(z) = \frac{1/p}{(1 - z)^{1+(1/p)} \log e/(1 - z)} - \frac{1}{(1 - z)^{1+(1/p)} (\log e/(1 - z))^2}. \]

We have
\[ |f'(z)|^p \leq \frac{1}{|1 - z|^{1+p}(\log |e/(1 - z)|)^p} \left( \frac{1}{p} + \frac{1}{\log e/(1 - z)} \right)^p \leq \left( \frac{1}{p} + \frac{1}{\log e/2} \right)^p \frac{1}{|1 - z|^{1+p}(\log |e/(1 - z)|)^p}. \]

Using Lemmas 2.4 and 2.5, we obtain that
\[ \int_0^{2\pi} \frac{1}{|1 - re^{i\theta}|^{1+p}(\log |e/(1 - re^{i\theta})|)^p} \, d\theta \]
\[ = \int_0^{2\pi} \frac{1}{|1 - re^{i\theta}|^{p/2}(\log |e/(1 - re^{i\theta})|)^p |1 - re^{i\theta}|^{1+(p/2)}} d\theta \]
\[ \leq \frac{C}{(1 - r)^{p/2} (\log e/(1 - r))^p} \int_0^{2\pi} \frac{1}{|1 - re^{i\theta}|^{1+(p/2)}} d\theta \]
\[ = \frac{C}{(1 - r)^p (\log e/(1 - r))^p}. \]

This shows that \( f(z) \in BH_{p,L} \), and the proof is complete. \( \square \)

We also note that \( BH_{p,L} \subset (H^p \cap \mathscr{D}_{p-1}) \) for every \( p \in (1, \infty) \) is proper. Indeed, we have the following theorem.

**Theorem 2.3.** Given \( p \) with \( 1 < p < \infty \), there exists a function \( f \) which belongs to \( (H^p \cap \mathscr{D}_{p-1}) \setminus BH_{p,L} \).

**Proof.** Let
\[ f(z) = \frac{1}{(1 - z)^{1/p} (\log(2e^{2\sqrt{p}})/(1 - z))^{1/\sqrt{p}}} , \quad z \in D. \]
Then
\[ \log \frac{2e^{2\sqrt{p}}}{1 - |z|} \sim \log \frac{1}{1 - |z|} \quad \text{as } |z| \to 1^- . \]
Take \( g_c \) in the proof of Theorem 2.2 with \( c = \sqrt{p} \). Then
\[ M_p^p(r, f) \sim M_1(r, g_{\sqrt{p}}), \quad r = |z| \to 1^- , \]
which implies \( f \in H^p \) for every \( 1 < p < \infty \).

As for proving \( f \in \mathscr{D}_{p-1} \) \((1 < p < \infty)\), it can be deduced by directly calculating
\[ \int_0^1 r(1 - r)^{p-1} M_p^p(r, f') dr < \infty . \]
Thus, we find \( f \in H^p \cap \mathscr{D}_{p-1} \).

On the other hand, using Lemma 2.4 yields
\[ \frac{1}{2\pi} \int_0^{2\pi} |f'(re^{i\theta})|^p d\theta \]
\[ \geq \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|1 - re^{i\theta}|^{1+p} \log(2e^{2\sqrt{p}})/(1 - re^{i\theta})^{\sqrt{p}}} \]
ON A LOGARITHMIC HARDY-BLOCH TYPE SPACE

\[
\times \left( \frac{1}{p} - \frac{1}{\sqrt{p}} \left| \log(2e^{2\sqrt{p}})/(1-re^{i\theta}) \right| \right)^p \, d\theta \\
\geq \frac{\delta}{2\pi(2p)^p} \frac{1}{(\log(2e^{2\sqrt{p}})/(1-r))^{1/p}} \frac{1}{\sqrt{p}} \int_0^{2\pi} \frac{1}{|1-re^{i\theta}|^{1+p}} \, d\theta \\
\geq \frac{\delta}{2\pi(2p)^p} \frac{1}{(1-r)^p(\log(2e^{2\sqrt{p}})/(1-r))^{1/p}},
\]

where \( \delta > 0 \) is a constant. Therefore,

\[
(1-r) \log \frac{e}{1-r} M_p(r, f') \geq \frac{\delta^{1/p}}{2p(2\pi)^{1/p}} \frac{\log e/1-r}{(\log(2e^{2\sqrt{p}})/(1-r))^{1/p}} \\
\rightarrow \infty, \quad r \rightarrow 1^-,
\]

which implies \( f \notin BH_{p,L} \), and this concludes the proof. \( \square \)

**Remark 2.1.** If we take

\[
f(z) = \frac{1}{(1-z)^{1/p} \log(2e^{2\sqrt{p}})/(1-z)}, \quad z \in D.
\]

Carefully checking the proofs of Theorems 2.2 and 2.3, we have

\[
M_p(r, f') \approx \frac{1}{(1-r) \log e/(1-r)}.
\]

In [17], the second author showed that \( \beta_L \subseteq VMOA \), the vanishing mean oscillation of analytic functions in \( D \). So we want to know whether \( BH_{p,L} \subseteq VMOA \) for \( p \in (0, \infty) \). However, the answer is negative. For more information for \( VMOA \) space, see [1, 4, 19].

**Theorem 2.4.** Suppose \( 0 < p < \infty \), then

\[
f(z) = \int_0^z \frac{1}{(1-t)^{1+1/p} \log e/(1-t)} \, dt \in BH_{p,L},
\]

but \( f(z) \notin VMOA \).

**Proof.** The proof for \( f \in BH_{p,L} \) is similar to Theorem 2.2, we omit the details here.
It is well known that the space
\[ VMOA \subset BMOA \subset B \bigcap \left( \bigcap_{0 < p < \infty} H^p \right). \]
We have
\[ f'(r) = \frac{1}{(1-r)^{1+1/p} \log e/(1-r)}, \]
and then
\[ (1-r)f'(r) = \frac{1}{(1-r)^{1/p} \log e/(1-r)}, \]
which tends to infinity when \( r \) tends to 1. Hence, \( f \notin VMOA \).

Our next objective is to estimate the growth of \( f \in BH_{p,L} \) \((0 < p < \infty)\). We begin with two lemmas.

**Lemma 2.7.** Given \( p \) with \( 1 \leq p < \infty \) and \( r \) with \( 0 < r < 1 \), then
\[
\int_0^r \frac{1}{(1-s)^{1+1/p} \log e/(1-s)} \, ds \leq 2p \left( 1 + pe^{2/p} \right) \frac{1}{(1-r)^{1/p} \log e/(1-r)}. 
\]

**Proof.** Let \( r_* = 1 - e^{1-2p} \). Using Lemma 2.3, we easily obtain
\[
\int_0^r \frac{1}{(1-s)^{1+1/p} \log e/(1-s)} \, ds \leq \int_0^{r_*} \frac{1}{(1-s)^{1+1/2p}} \, ds 
\leq 2pe^{(2p-1)/2p},
\]
if \( r \leq r_* \). When \( r > r_* \), we have
\[
\int_0^r \frac{1}{(1-s)^{1+1/p} \log e/(1-s)} \, ds 
= \int_0^{r_*} \frac{1}{(1-s)^{1+1/p} \log e/(1-s)} \, ds + \int_{r_*}^r \frac{1}{(1-s)^{1+1/2p}} \, ds 
\leq 2pe^{(2p-1)/2p} + \frac{1}{(1-r)^{1+1/2p} \log e/(1-r)} \int_0^r \frac{1}{(1-s)^{1+1/2p}} \, ds 
\leq 2pe^{(2p-1)/2p} + 2pe^{2/p} \frac{1}{(1-r)^{1/p} \log e/(1-r)}. 
\]
Hence, (2) holds. \qed
Lemma 2.8. [8] Any function $f \in H^p$ ($0 < p < \infty$) can be expressed in the form $f(z) = f_1(z) + f_2(z)$, where $f_1$ and $f_2$ are nonvanishing $H^p$ functions such that $\|f_n\|_{H^p} \leq 2\|f\|_{H^p}$, $n = 1, 2$.

Theorem 2.5. If $f \in BH_{p,L}$, $(0 < p < \infty)$, then

$$|f(z)| \leq \frac{C\|f\|_{p,L}}{(1-r)^{1/p} \log e/(1-r)}, \quad r = |z|,$$

where

$$C = \begin{cases} 
4\pi p(p-1)^{(p-1)/p} \cdot (1 + p \cdot e^{1/2p}), & 1 < p < \infty, \\
8(1 + e^{1/2}), & p = 1, \\
2^{2+(3/p)-3p}/(1-p), & 0 < p < 1.
\end{cases}$$

Proof. For $r$ with $0 < r < 1$, we have

$$f(re^{i\theta}) = \int_0^r f'(se^{i\theta})e^{i\theta} ds + f(0).$$

We set $\rho = (1+s)/2$, by Cauchy formula

$$f'(se^{i\theta}) = \frac{1}{2\pi i} \int_{|\zeta|=\rho} \frac{f'()}{\zeta - se^{i\theta}} d\zeta = \frac{\rho}{2\pi} \int_0^{2\pi} f'(\rho e^{i(t+\theta)})e^{i(t-\theta)} \frac{dt}{\rho e^{it} - s}.$$

Case I: $1 < p < \infty$. Let $q$ be the conjugate index of $p : (1/p) + (1/q) = 1$, then, by Hölder’s inequality, Lemmas 2.4 and 2.7, we give

$$|f(re^{i\theta})| \leq |f(0)| + \frac{1}{2\pi} \int_0^r \int_0^{2\pi} \frac{|f'(\rho e^{i(t+\theta)})e^{i(t-\theta)}|}{|\rho e^{it} - s|} dt ds \\ \leq |f(0)| + \left(\frac{1}{2\pi}\right)^{1/q} \cdot \left(\frac{4\pi q}{q - 1}\right)^{1/q} \int_0^r M_p(p, f') \cdot \frac{1}{(1-s)^{1/p}} ds \\ \leq |f(0)| + 4\pi p(p-1)^{(p-1)/p} \left(1 + p \cdot e^{1/2p}\right) \\ \cdot \frac{\|f\|_{p,L}}{(1-r)^{1/p} \log e/(1-r)} \\ \leq 4\pi p(p-1)^{(p-1)/p} \left(1 + p \cdot e^{1/2p}\right) \frac{\|f\|_{p,L}}{(1-r)^{1/p} \log e/(1-r)}.$$
Case II: \( p = 1 \). Using Lemma 2.7 again yields

\[
|f(re^{i\theta})| \leq |f(0)| + \frac{1}{2\pi} \int_0^r \int_0^{2\pi} \left| \frac{f'(pe^{i(t+\theta)})e^{i(t-\theta)}}{|pe^{it} - s|} \right| dt \, ds
\]

\[
= |f(0)| + 4 \int_0^r \frac{1}{(1-s)^2 \log e/(1-s)} \, ds \cdot ||f||_{p,L}
\]

\[
\leq 8(1 + e^{1/2}) \frac{||f||_{p,L}}{(1-r) \log e/(1-r)}.
\]

Case III: \( 0 < p < 1 \). If \( f'(z) \neq 0 \) in \( z \in D \), then the function \( F(z) = (f'(z))^p \) is analytic, and \( f \in \mathcal{BH}_{p,L} \) gives

\[
M_1(r, F) = \{M_p(r, f')\}^p \leq \frac{||f||_{p,L}^p}{(1-r)^{1/p} \log e/(1-r)^p}.
\]

By the Cauchy formula, we find

\[
|F(re^{i\theta})| = \left| \frac{\rho}{2\pi} \int_0^{2\pi} F(\rho e^{it}) e^{it} dt \right| \leq \frac{2||f||_{p,L}^p}{(1-r)^{1+p} \log e/(1-r)^p},
\]

which implies that

\[
|f'(re^{i\theta})| \leq \frac{\sqrt{2}||f||_{p,L}^p}{(1-r)^{1+1/p} \log e/(1-r)}.
\]

Then

\[
M_1(r, f') = \frac{1}{2\pi} \int_0^{2\pi} |f'(re^{i\theta})|^{1-p} |f'(re^{i\theta})|^p d\theta
\]

\[
\leq \{M_\infty(r, f')\}^{1-p} \{M_p(r, f')\}^p
\]

\[
\leq \frac{2^{(1-p)/p}}{(1-r)^{1/p} \log e/(1-r)} \cdot ||f||_{p,L}^p.
\]

We deduce

\[
|f(re^{i\theta})| \leq |f(0)| + \frac{1}{2\pi} \int_0^r \int_0^{2\pi} \left| \frac{f'(pe^{i(t+\theta)})e^{i(t-\theta)}}{|pe^{it} - s|} \right| dt \, ds
\]

\[
\leq |f(0)| + 2^{2/p} \frac{p}{1-p} \cdot \frac{||f||_{p,L}}{(1-r)^{1/p} \log e/(1-r)}
\]

\[
\leq \frac{2^{2/p}}{1-p} \cdot \frac{||f||_{p,L}}{(1-r)^{1/p} \log e/(1-r)}.
\]
If \( f'(z) \) has zeros, we fix \( R < 1 \) and use Lemma 2.8 to write
\[
f'(Rz) = f'_1(z) + f'_2(z),
\]
where \( f_1 \) and \( f_2 \) do not vanish and
\[
\|f'_n\|_{H^p} \leq 2M_p(R, f') \leq \frac{2\|f\|_{p,L}}{(1 - R) \log e/(1 - R)}, \quad n = 1, 2.
\]
Since \( f'_n(z) \neq 0 \) \((n = 1, 2)\), it follows that
\[
|f'(R^2 e^{i\theta})| \leq |f'_1(Re^{i\theta})| + |f'_2(Re^{i\theta})| \leq \frac{2^{2+1/p}\|f\|_{p,L}}{(1 - R)^{1+1/p} \log e/(1 - R)}.
\]
Then
\[
|f'(re^{i\theta})| \leq \frac{2^{3+2/p}\|f\|_{p,L}}{(1 - r)^{1+1/p} \log e/(1 - r)},
\]
which implies that
\[
|f(re^{i\theta})| \leq \frac{2^{2+3/p-3p}}{1-p} \cdot \frac{\|f\|_{p,L}}{(1 - r)^{1/p} \log e/(1 - r)}.
\]
This completes the proof of Theorem 2.5. \(\square\)

For \( 0 < p < \infty \), we set
\[
H^\infty_{p,L} = \left\{ f \in H(D), \; |f(z)| = \sup \left( \frac{1}{(1 - r)^{1/p} \log e/(1 - r)} \right) < \infty, \; |z| = r \right\}.
\]
Then \( BH_{p,L} \) and the classic Bloch space \( \mathcal{B}_1 \) are included in \( H^\infty_{p,L} \). It turns out that neither \( H^\infty_{p,L} \) nor \( H^\infty_{p,L} \) is contained in \( H^p \) for every \( 1 < p < \infty \). Indeed,
\[
f(z) = \sum_{n=0}^{\infty} z^{2^n}, \quad z \in D
\]
is in \( \mathcal{B}_1 \), which implies that \( f \in H^\infty_{p,L} \), but not in \( H^p \). On the other hand, taking the function \( f(z) \) in Theorem 2.4, we find that \( f \in H^p \) but not \( f \notin H^\infty_{p,L} \).

We set \( \mathcal{U} \) here to be the class of all univalent functions in \( D \). Then Prawitz, see [13, page 17], deduced the following theorem.
Theorem D. Suppose that $0 < p < \infty$. If $f \in \mathcal{U}$ and $\int_0^1 M_p^\infty(r, f)\,dr < \infty$, then $f \in H^p$. Using this result, we can give the following theorem.

Theorem 2.6. Suppose that $1 < p < \infty$ and $f \in \mathcal{U} \cap H^\infty_{p,L}$. Then $f \in H^p$.

Acknowledgments. We thank the referee for numerous stylistic corrections.

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