SEMIGROUP COMPACTIFICATIONS OF ZAPPA PRODUCTS

H.D. JUNGHENN AND P. MILNES

ABSTRACT. A group $G$ with subgroups $S$ and $T$ satisfying $G = ST$ and $S \cap T = \{e\}$ gives rise to functions $[t,s] \in S$ and $\langle t,s \rangle \in T$ such that $(st)(s't') = (s[t,s'])(\langle t,s' \rangle t')$. This notion may be extended to arbitrary semigroups $S, T$ with identities, producing the Zappa product of $S$ and $T$, a generalization of direct and semidirect product. Necessary and sufficient conditions are given for a semigroup compactification of a Zappa product $G$ of topological semigroups $S$ and $T$ to be canonically isomorphic to a Zappa product of compactifications of $S$ and $T$. The result is applied to various types of compactifications of $G$, including the weakly almost periodic and almost periodic compactifications.

1. Introduction. Let $G$ be a semigroup with identity $e$, and let $S$ and $T$ be subsemigroups containing $e$ such that every member of $G$ is uniquely expressible as a product $st$ with $s \in S$ and $t \in T$. It follows that there exist functions $[\cdot, \cdot] : T \times S \to S$ and $\langle \cdot, \cdot \rangle : T \times S \to T$ such that

$$ts = [t,s]\langle t,s \rangle, \quad s \in S, \ t \in T.$$ 

Identifying $G$ with $S \times T$, we see that multiplication in $G$ may be expressed as

$$ (s,t)(s',t') = (s[t,s'], \langle t,s \rangle t'). $$

(1)

Associativity and the identity property imply the following relations:

(2) \quad $[t,e] = e, \quad [e,s] = s, \quad \langle e,s \rangle = e, \quad \langle t,e \rangle = t,$

(3) \quad $[tt',s] = [t,[t',s]], \quad [t,ss'] = [t,s]\langle [t,s],s' \rangle$

(4) \quad $\langle t,ss' \rangle = \langle \langle t,s \rangle, s' \rangle \rangle, \quad \text{and} \quad \langle tt',s \rangle = \langle t,\langle t',s \rangle \rangle \langle t',s \rangle.$

Conversely, if $S$ and $T$ are semigroups with identities $e$ and $[\cdot, \cdot]$ and $\langle \cdot, \cdot \rangle$ are mappings that satisfy (2)–(4), then $G := S \times T$ is a semigroup with identity $(e,e)$ under multiplication given by (1). $G$ is then called

Received by the editors on June 27, 2012, and in revised form on October 18, 2012.

DOI:10.1216/RMJ-2014-44-6-1903 Copyright ©2014 Rocky Mountain Mathematics Consortium
a Zappa product of $S$ and $T$, which we denote by $G = S \times_z T$. If $S$ and $T$ are groups, then $G$ is a group and

$$(s, t)^{-1} = (e, t^{-1})(s^{-1}, e) = ([t^{-1}, s^{-1}], [t^{-1}, s^{-1}]).$$

$S \times_z T$ is a semidirect product if either of the Zappa product mappings $[\cdot, \cdot]$ or $\langle \cdot, \cdot \rangle$ is trivial (i.e., $[t, \cdot]$ or $\langle \cdot, s \rangle$ is the identity mapping), and a direct product if both are trivial. Zappa products of groups were first studied in [10] and subsequently in, for example, [8, 9]. The semigroup case was considered in [6] in the context of finite automata.

Our goal in this paper is to give necessary and sufficient conditions for a compactification of a topological Zappa product to be a Zappa product of compactifications. Our results complement and extend those of [3, 4, 5, 7], which consider the direct and semidirect product cases. (See also [1] for a summary.)

2. Semigroup compactifications. In this section, we give a brief overview of those aspects of the theory of semigroup compactifications that will be needed in the sequel. For details, the reader is referred to [1].

Let $G$ be a semitopological semigroup with identity, $C(G)$ the space of bounded, continuous, complex-valued functions on $G$, and $L(s)$, respectively, $R(s)$, the left, respectively, right, translation operator on $C(G)$. A (right topological) compactification of $G$ is a pair $(\psi, G')$, where $G'$ is a compact, Hausdorff, right topological semigroup and $\psi : G \mapsto G'$, the compactification map, is a continuous homomorphism with dense range such that the mappings $x \mapsto \psi(s)x$ are continuous. It follows that $F := \psi^*(C(G'))$ is a translation invariant $C^*$subalgebra of $C(G)$ with the property that, for $x \in GF$, the spectrum of $F$, $x_\ell(F) \subseteq F$, where $x_\ell(f)(s) := x(L(s)f)$. We shall call such an algebra $m$-admissible. Note that $\{x_\ell(f) \mid x \in GF\} = \overline{R(G)f}$ is the closure of $R(G)f$ in the topology of pointwise convergence on $C(G)$. Conversely, if $F \subseteq C(G)$ is $m$-admissible, then $GF$ is a compactification of $G$, where the compactification map $\varepsilon_F : G \mapsto GF$ is evaluation. The pair $(\varepsilon_F, GF)$ is called the (canonical) $F$-compactification of $G$. (See [1, pages 108–109].)

A compactification $(\psi, G')$ of $G$ is an extension of a compactification $(\theta, G'')$ and $(\theta, G'')$ is a factor of $(\psi, G')$, written $(\theta, G'') \preceq (\psi, G')$, if
there exists a continuous map $\pi : G' \to G''$, the canonical homomorphism, such that $\pi \circ \psi = \theta$. If $\pi$ is one-to-one, then $\pi$ is an isomorphism and $(\psi, G')$ and $(\theta, G'')$ are said to be (canonically) isomorphic, written $(\psi, G') \cong (\theta, G'')$ or simply $G' \cong G''$. Every compactification $(\psi, G')$ of $G$ is isomorphic to the $F$-compactification $(\varepsilon_F, G^F)$, where $F = \psi^*(\mathbb{C}(G'))$. The relation $\leq$ partially orders the collection of (equivalence classes of) compactifications of $G$ rendering it a complete lattice with upper bound $G^{LC}$ (see below) and lower bound the trivial compactification $\{\varepsilon(e)\}$.

We shall frequently make use of the following fact: If $\varphi : G_1 \to G$ is a continuous function from a semitopological semigroup $G_1$ into $G$ and if $F \subseteq \mathbb{C}(G)$ and $F_1 \subseteq \mathbb{C}(G_1)$ are $m$-admissible with $\varphi^*(F) \subseteq F_1$, then $\varphi$ has an extension $\overline{\varphi}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
G_1^{F_1} & \xrightarrow{\overline{\varphi}} & G^F \\
\varepsilon_{F_1} \uparrow & & \uparrow \varepsilon_F \\
G_1 & \xrightarrow{\varphi} & G
\end{array}
$$

The map $\overline{\varphi}(x)$ is defined by $\overline{\varphi}(x)(f) = x(\varphi^*(f))$ and is a homomorphism if and only if $\varphi$ is.

A compactification of $G$ possessing a given property $\mathcal{P}$ is called the universal $\mathcal{P}$-compactification of $G$ if it is an extension of every compactification with property $\mathcal{P}$ [1, page 115]. Here are some examples of $F$-compactifications and their universal properties.

- **Left multiplicatively continuous functions.** $F = LMC(G)$ is the algebra of all $f \in \mathbb{C}(G)$ such that $s \mapsto x(L(s)f)$ is continuous for each multiplicative mean $x$. (Equivalently, $R(S)f$ is relatively compact in the topology of pointwise convergence on $S$.) $G^{LMC}$ is the universal right topological semigroup compactification of $G$.

- **Left continuous functions.** $F = LC(G)$ is the algebra of all $f \in \mathbb{C}(G)$ such that $s \mapsto L(s)f$ is norm continuous. $G^{LC}$ is the right topological semigroup compactification of $G$ that is universal with respect to the property that the mapping $(s, x) \mapsto \varepsilon_F(s)x$ from $G \times G^F$ to $G^F$ is continuous. This implies that $G^F$ has the restricted joint continuity property, namely, that multiplication on $\varepsilon_F(K) \times G^{LC}$ is continuous for
each compact set $K \subset G$.

- **Left continuous functions on compacta.** $F = K(G)$ is the algebra of all $f \in LMC(G)$ such that $R(S)f$ is relatively compact in the topology of uniform convergence on compacta (equivalently: the restriction of $s \mapsto L(s)f$ to compact sets is norm continuous). $G^K$ is the right topological semigroup compactification of $G$ that is universal with respect to the restricted joint continuity property described in the preceding example.

- **Distal functions.** $F = D(G)$ is the algebra of all $f \in LMC(G)$ such that $(uvw)(f) = (uw)(f)$ for all $u, v^2 = v, w \in G^{LMC}$. $G^D$ is the universal right topological group compactification of $G$.

- **Weakly almost periodic functions.** $F = WAP(G)$ is the algebra of all functions $f \in C(G)$ such that $R(G)f$ (equivalently, $L(G)f$) is relatively weakly compact. $G^{WAP}$ is the universal semitopological semigroup compactification of $G$.

- **Almost periodic functions.** $F = AP(G)$ is the algebra of all $f \in C(G)$ such that $R(G)f$ (equivalently, $L(G)f$) is relatively norm compact. $G^{AP}$ is the universal topological semigroup compactification of $G$.

- **Strongly almost periodic functions.** $F = SAP(G)$ is the algebra generated by the finite dimensional unitary subspaces of $C(G)$. $G^{SAP}$ is the universal topological group compactification of $G$.

By Ellis’s theorem on separate and joint continuity of group actions, $SAP(G) = WAP(G) \cap D(G)$.

We also define

$$LCWAP(G) := LC(G) \cap WAP(G)$$

and

$$KWAP(G) := K(G) \cap WAP(G).$$

It is clear from the inclusion relations among the function spaces that $G^{SAP} \lesssim G^{AP} \lesssim G^{WAP}$ and $G^{LC} \lesssim G^K$. Moreover, if $G$ contains a dense group, then $G^{SAP} \cong G^{AP}$; if $G$ is a $K$-space or is metrizable, then $G^{LC} \cong G^K$; and if $G$ is a topological group, then $G^{WAP} \lesssim G^{LC}$. 

For more details about these spaces the reader is referred to [1, Chapter 4].

3. A general compactification theorem. For the remainder of the paper we assume that $S$ and $T$ are topological semigroups and that the Zappa product mappings of $G = S \times T$ are jointly continuous, so that $G$ is a topological semigroup.

Given an $m$-admissible algebra $F \subseteq C(G)$, we seek conditions under which $G^F$ is (canonically isomorphic to) a Zappa product of compactifications $(\theta, S')$ of $S$ and $(\psi, T')$ of $T$. This means that the compactification map of $S' \times T'$ is the product map $\theta \times \psi$. In this case, we write $G^F \cong S' \times T'$. Note that we then have

$$([\psi(t), \theta(s)], (\psi(t), \theta(s))) = (\theta(e), \psi(t))(\theta(s), \psi(e))$$

$$= (\theta \times \psi)((e, t)(s, e))$$

$$= (\theta([t, s]), \psi([t, s])), $$

i.e., the Zappa product maps on $S' \times T'$ are extensions of the corresponding maps on $S \times T$. This implies that if $G$ is a semidirect (direct) product then $S' \times T'$ is a semidirect (direct) product.

Theorem 3.3 below is the basis for the compactification results in this paper. The corresponding theorem for the special case of semidirect products was proved in [5] using results on tensor products. We give a direct, self-contained proof based on Lemmas 3.1 and 3.2. The following notation will be convenient:

$$q_S : S \hookrightarrow S \times T, \quad s \mapsto (s, e);$$

$$q_T : T \hookrightarrow S \times T, \quad t \mapsto (e, t);$$

$$p_S : S \times T \twoheadrightarrow S, \quad (s, t) \mapsto s;$$

$$p_T : S \times T \twoheadrightarrow T, \quad (s, t) \mapsto t;$$

$$r_S = q_S \circ p_S; \quad r_T = q_T \circ p_T.$$

Lemma 3.1. Let $G = S \times T$ have a multiplication relative to which it is a semigroup with identity $(e, e)$. Then $G$ is a Zappa product if and only if the mappings $q_S$ and $q_T$ are homomorphisms and $r_S \cdot r_T = \text{id}_G$. If $G$ is a Zappa product, then it is a semidirect product if and only if
either \( p_S \) and \( p_T \) is a homomorphism, and a direct product if and only if both maps are homomorphisms.

**Proof.** The conditions on the maps \( q \) and \( r \) reduce to the identities

\[
(s, e)(s', e) = (ss', e), \quad (e, t)(e, t') = (e, tt'), \quad \text{and} \quad (s, e)(e, t) = (s, t).
\]

For the sufficiency of the first assertion, define

\[
[t, s] = p_S((e, t)(s, e)) \quad \text{and} \quad \langle t, s \rangle = p_T((e, t)(s, e))
\]

so that \( (e, t)(s, e) = ([t, s], \langle t, s \rangle) \) It is straightforward to check that identities (2)–(4) in the introduction are satisfied. For example, \( (e, t) = (e, t)(e, e) = ([t, e], \langle t, e \rangle) \) implies that \([t, e] = e\) and \(\langle t, e \rangle = t\), and the calculations

\[
([tt', s], \langle tt', s \rangle) = (e, tt')(s, e) = (e, t)(t', s))
\]

\[
= ([t, [t', s]], \langle t, [t', s] \rangle)(e, \langle t', s \rangle)
\]

\[
= ([t, [t', s]], e)(e, [t', s])(e, \langle t', s \rangle)
\]

\[
= ([t, [t', s]], e)(e, [t', s])\langle t', s \rangle)
\]

\[
= ([t, [t', s]], \langle t, [t', s] \rangle\langle t', s \rangle)
\]

show that \([tt', s] = [t, [t', s]]\) and \(\langle tt', s \rangle = \langle t, [t', s] \rangle\langle t', s \rangle\). The necessity and the remaining assertions of the lemma are also straightforward. \(\square\)

**Lemma 3.2.** Let \( X \) and \( Y \) be topological spaces with \( X' \) dense in \( X \). Suppose that \( f \) is a bounded, complex-valued function on \( X \times Y \) such that \( f(X', \cdot) \) is relatively compact in \( C(Y) \) and \( f(\cdot, Y) \subseteq C(X) \). Then \( f(X, \cdot) \) is relatively compact in \( C(Y) \), \( f \) is jointly continuous, and \( x \mapsto f(x, \cdot) \) is continuous.

**Proof.** Set \( K = \overline{f(X', \cdot)} \subseteq C(Y) \), and let \( x \in X \) and \( x'_\alpha \in X' \) with \( x'_\alpha \rightarrow x \). Taking a subnet if necessary, we may assume that \( f(x'_\alpha, \cdot) \rightarrow g \in K \). Given \( \epsilon > 0 \) choose \( \alpha_0 \) such that

\[
|f(x'_\alpha, y) - g(y)| < \epsilon \quad \text{for all} \quad y \in Y \quad \text{and} \quad \alpha \geq \alpha_0.
\]

Since \( f(\cdot, y) \) is continuous, taking limits we see that \( g(y) = f(x, y) \). Therefore, \( f(x, \cdot) \in K \); hence, \( f(X, \cdot) \) is relatively compact in \( C(Y) \).
Now let \((x_\alpha, y_\alpha) \rightarrow (x_0, y_0)\) in \(X \times Y\). By the result of the first paragraph, we may assume that \(f(x_\alpha, \cdot) \rightarrow h \in C(Y)\). As above, given \(\varepsilon > 0\), there exists \(\alpha_0\) such that

\[
|f(x_\alpha, y) - h(y)| < \varepsilon \quad \text{for all} \quad \alpha \geq \alpha_0 \quad \text{and} \quad y \in Y.
\]

It follows that \(h(y) = f(x_0, y)\) and hence \(f(x_\alpha, \cdot) \rightarrow f(x_0, \cdot)\). Since \(h\) is continuous and \(y_\alpha \rightarrow y_0\), it follows easily that \(\{f(x_\alpha, y_\alpha)\}\) is a Cauchy net and hence converges to some \(L \in \mathbb{C}\). From (5) we see that \(L = h(y_0)\). Therefore, \(L = f(x_0, y_0)\) and \(f(x_\alpha, y_\alpha) \rightarrow f(x_0, y_0)\).

\[
\text{Theorem 3.3. Let } F \subseteq C(G) \text{ be m-admissible. Then } G^F \cong S' \times_S T' \text{ for some compactifications of } S \text{ and } T \text{ if and only if (a) } r^*_S(F) \cup r^*_T(F) \subseteq F \text{ and (b) for each } f \in F, \text{ either } f(S, \cdot) \text{ is relatively compact in } C(T) \text{ or } f(\cdot, T) \text{ is relatively compact in } C(S). \text{ If (a) and (b) hold, then } S' \cong S_A \text{ and } T' \cong T_B, \text{ where } A := q^*_S(F) \text{ and } B := q^*_T(F). \]

\[
\text{Proof. For the necessity, suppose that } G^F \cong S' \times_S T' \text{ for some compactifications } (\theta, S') \text{ of } S \text{ and } (\psi, T') \text{ of } T, \text{ and let } \varphi : G^F \hookrightarrow S' \times_S T' \text{ denote the compactification isomorphism. Given } f \in F, \text{ choose } g \in C(S' \times_S T') \text{ such that } f = \varepsilon^*_F \circ \varphi^*(g). \text{ Since the mapping } x \mapsto g(x, \cdot) : S' \hookrightarrow C(T') \text{ is norm continuous, } g(S', \cdot) \text{ is norm compact in } C(T'). \text{ Since } f(s, t) = g(\theta(s), \psi(t)), \text{ we have } f(\cdot, T) \text{ is norm compact in } C(S). \text{ Setting } h := r^*_S(g), \text{ we have}
\]

\[
\varepsilon^*_F \circ \varphi^*(h)(s, t) = g(\theta(s), \psi(e)) = f(s, e),
\]

so \(r^*_S(f) = \varepsilon^*_F \circ \varphi^*(h) \in F\). Similarly, \(r^*_T(f) \in F\). Thus, conditions (a) and (b) hold.

We now have the commutative diagram

\[
\begin{array}{cccccccc}
S^A & \xrightarrow{\bar{q}_S} & G^F & \xrightarrow{\varphi} & S' \times_S T' & \xrightarrow{p_{S'}} & S' \\
\uparrow & & \uparrow & & \uparrow & & \\
S & \xrightarrow{q_S} & G & \xrightarrow{id} & G & \xrightarrow{p_S} & S \\
\end{array}
\]

with a similar diagram for \(T\). Set \(\gamma = p_{S'} \circ \varphi \circ \bar{q}_S : S^A \rightarrow S'. \text{ Since } \gamma \circ \varepsilon_A = \theta \circ p_S \circ q_S = \theta, \gamma \text{ is a compactification homomorphism. To see that } \gamma \text{ is one-to-one and hence an isomorphism, note first that}
\]

\[
(\gamma(\varepsilon_A(s)), \psi(e)) = (\theta \times \psi) \circ q_S(s) = \varphi \circ \bar{q}_S(\varepsilon_A(s));
\]
Then so that, by Lemma 3.2, the function $e$ is jointly continuous. Moreover, $\theta_S$ is a group isomorphism. Since $f \in \mathbb{C}^0(G_f)$, $h \in \mathbb{C}^0(S, T)$, and therefore $\phi_S, \phi_T$ are one-to-one and hence a homeomorphism. Let $\tilde{f} \in C(G_f)$ and $f = \varepsilon_p(\tilde{f})$. Assume without loss of generality that $K := \tilde{f}(S, \cdot)$ is compact in $C(T)$. Since $f(s, t) = q_s^*(R(e, t)f)(s) = q_t^*(L(s, e)f)(t)$,

$f(\cdot, t) \in A$ and $K \subseteq B$. Define $\tilde{f}$ on $S^A \times T$ by $\tilde{f}(x, t) = x(f(\cdot, t))$ and note that $\tilde{f}(\cdot, t)$ is continuous for each $t$ and $\bar{f}(\varepsilon_A(s), \cdot) = f(s, \cdot) \in K$. By Lemma 3.2, $\tilde{f}$ is jointly continuous, $x \mapsto \tilde{f}(x, \cdot)$ is continuous and $\tilde{f}(S^A, \cdot) \subseteq K$. For $(x, y) \in S^A \times T^B$, define $\tilde{f}(x, y) = y(\tilde{f}(x, \cdot))$. Then $\tilde{f}$ is separately continuous and $x \mapsto \tilde{f}(x, \cdot)$ is norm continuous so that, by Lemma 3.2, the function $\tilde{f}$ is jointly continuous. Moreover, $\tilde{f}(\varepsilon_A(s), \varepsilon_B(t)) = \tilde{f}(\varepsilon_A(s), t) = f(s, t)$, hence $\pi^* \tilde{f} = \tilde{f}$, verifying the claim.

Now give $S^A \times T^B$ the unique multiplication that makes $\pi$ a semigroup isomorphism. Since $\pi \circ \varepsilon_p = \pi$, $\pi$ is a compactification isomorphism. The diagram above now holds with $\varphi = \pi$, $(\theta, S') = (\varepsilon_A, S^A)$ and $(\psi, T') = (\varepsilon_B, T^B)$. Note that

$$
\pi^{-1}(\varepsilon_A(s), \varepsilon_B(t)) = \varepsilon_p(s, e) \cdot \varepsilon_p(e, t) = \varphi_S(\varepsilon_A(s)) \cdot \varphi_T(\varepsilon_B(t));
$$

hence $\pi^{-1}(x, y) = \varphi_S(x) \cdot \varphi_T(y)$, and therefore $(x, y) = (\pi \circ \varphi_S(x)) \cdot (\pi \circ \varphi_T(y))$. But, for $s \in S$, $\pi(\varphi_S(\varepsilon_A(s))) = \pi(\varepsilon_p(s, e)) = (\varepsilon_A(s), \varepsilon_B(e)) = q_{sA}(\varepsilon_A(s))$; hence, $\pi \circ \varphi_S(x) = q_{sA}(x) = r_{sA}(x, y)$ for all $(x, y) \in S^A \times T^B$. A similar identity holds for $\pi \circ \varphi_T$. Therefore, $q_{sA}$ and
are homomorphisms and \( r_{SA} \cdot r_{TB} \) is the identity mapping. That \( S^A \times T^B \) is a Zappa product now follows from Lemma 3.1.

**Corollary 3.4.** The maximal factor and the minimal extension of a family \( \mathcal{G} := \{(\psi_i, G^{(i)}) \mid i \in I\} \) of Zappa product compactifications of \( G \) are Zappa product compactifications of \( G \). Thus, the collection of Zappa compactifications of \( G \) is a complete sublattice of the lattice of all right topological semigroup compactifications of \( G \).

*Proof.* The maximal factor of the family \( \mathcal{G} \) is \( G^H \), where \( H = \bigcap_i F_i \). Clearly, \( H \) satisfies conditions (a) and (b) of the theorem; hence, \( G^H \) is a Zappa product.

The minimal extension of \( \mathcal{G} \) is \( G^F \), where \( F \) is the intersection of all \( m \)-admissible algebras containing \( K := \bigcup_i \psi^* C(G^{(i)}) \). Let \( F' \) denote the set of all \( f \in F \) such that \( r^*_A(g), r^*_B(g) \in F \) and \( g(S, \cdot) \) is relatively compact for every \( g \in L(G)\overline{R(G)}^p f \). Then \( F' \) is an \( m \)-admissible algebra containing \( K \) and therefore equals \( F \). By the theorem, then, \( G^F \) is a Zappa product. \( \square \)

### 4. LC and WAP compactifications of \( G \).

**Theorem 4.1.** Let \( S \) be a compact topological group, \( T \) a topological semigroup, and let \( F = LC(G) \) or \( F = K(G) \). Then \( G^F \cong S' \times T' \) for some compactification \( S' \preceq S^{SAP} \) of \( S \) and compactification \( T' \preceq T^{Fr} \) of \( T \), where \( F_T \) is the corresponding space of functions on \( T \).

*Proof.* Let \( f \in F \) and \( \hat{f} \in C(G^F) \) with \( \varepsilon_p(\hat{f}) = f \). Since \( S \) is compact and the mapping \( s \mapsto L(s, e)f \) is norm continuous, condition (b) of Theorem 3.3 holds. Thus, it remains to show that \( r^*_S(f) \in F \) and \( r^*_T(f) \in F \). We prove this for \( F = LC(G) \), the proof for \( K(G) \) being similar.

Let \( x_\alpha := (u_\alpha, v_\alpha) \rightarrow x := (u, v) \) in \( G \). By the joint continuity of \([\cdot, \cdot]\) and the compactness of \( S \), \([v_\alpha, s] \rightarrow [v, s] \) uniformly in \( s \in S \). It follows that \( L(x_\alpha)r^*_Sf(s, t) = f(u_\alpha[v_\alpha, s], e) \) converges uniformly in \((s, t)\), so \( r^*_Sf \in LC(G) \).
Since \( r^*_S(F) \subset F \), \( r_S \) has a continuous extension \( \bar{\tau}_S : G^F \mapsto G^F \). From \( r_S \cdot r_T = \text{id}_G \) we then have
\[
L(x_\alpha) r^*_T f(s, t) = \hat{f} \left( \{ \bar{\tau}_S(\varepsilon(x_\alpha)\varepsilon(s, t)) \}^{-1} \varepsilon(x_\alpha)\varepsilon(s, t) \right),
\]
the inverse taken in the compact group \( H := \varepsilon(S \times \{ e \}) \). We claim that \( L(x_\alpha) r^*_T f(s, t) \) converges uniformly in \((s, t)\) to \( L(x) r^*_T f(s, t) \). If not, then there exist \( r > 0 \), a subnet \( \{ x_\beta \} \), and a net \( \{ y_\beta := (s_\beta, t_\beta) \} \) with \( \varepsilon(y_\beta) \to b \in G^{LC} \) such that, for all \( \beta \),
\[
\left| \hat{f} \left( \{ \bar{\tau}_S(\varepsilon(x_\beta)\varepsilon(y_\beta)) \}^{-1} \varepsilon(x_\beta)\varepsilon(y_\beta) \right) - \hat{f} \left( \{ \bar{\tau}_S(\varepsilon(x)\varepsilon(y_\beta)) \}^{-1} \varepsilon(x)\varepsilon(y_\beta) \right) \right| \geq r.
\]
But this is impossible because of the restricted joint continuity property of multiplication in \( G^F \), since the terms in braces converge in \( H \subseteq \varepsilon(G^F) \) and \( \varepsilon(x_\beta)\varepsilon(y_\beta), \varepsilon(x)\varepsilon(y_\beta) \to \varepsilon(x)b \). Therefore, \( r^*_T f \in LC(G) \).

That \( T' \subseteq T^{F_T} \) follows from the inclusion \( q^*_T(F) \subseteq F_T \), this because \( q_T : T \mapsto G \) is a continuous homomorphism. \( \square \)

**Theorem 4.2.** Let \( S \) be a compact topological group, \( T \) a topological semigroup, and let \( F = LCWAP(G) \) or \( F = KWAP(G) \). Then \( G^F \cong S' \times_z T' \) for some compactifications \( S' \) of \( S \) and \( T' \) of \( T \) if and only if \( r^*_S(F) \subseteq F \). In this case, \( S' \leq S^{SAP} \) and \( T' \leq T^{F_T} \), where \( F_T \) is the corresponding space of functions on \( T \).

**Proof.** The necessity is clear. For the sufficiency, let \( f \in F \) and \( \hat{f} \in C(G^F) \) with \( \varepsilon_f(\hat{f}) = f \). Since \( S \) is compact and the mapping \( s \mapsto L(s, e)f \) is norm continuous, condition (b) of Theorem 3.3 holds. Thus, it suffices to show that if \( r^*_S(f) \in F \Rightarrow r^*_T(f) \in F \). We do this for the case \( F = LCWAP(G) \), the other case being similar.

From Theorem 4.1, \( r^*_T(f) \in LC(G) \). To show that \( r^*_T(f) \in WAP(G) \) we use Grothendieck’s criterion. Let \( \{(s_m, t_m)\} \) and \( \{(s_n, t_n)\} \) be sequences in \( G \) such that the limits
\[
\ell_1 = \lim_m \lim_n f \circ r_T((s_n, t_n)(s_m, t_m))
\]
and
\[ \ell_2 = \lim_n \lim_m f \circ r_T((s_n,t_n)(s_m,t_m)) \]
exist. We need to show that \( \ell_1 = \ell_2 \). Set \( \varepsilon = \varepsilon_F \). By hypothesis, \( r_S \) has an extension \( \tau_S : GF \to GF \). Thus,
\[ (6) \quad \varepsilon \circ r_S((s_n,t_n)(s_m,t_m)) = \varepsilon(s_n[t_n,s_m],e) = \tau_S(\varepsilon(s_n,t_n)\varepsilon(s_m,t_m)) \]
and
\[ (7) \quad \varepsilon \circ r_T((s_n,t_n)(s_m,t_m)) \]
\[ = \{
\tau_S(\varepsilon(s_n,t_n)\varepsilon(s_m,t_m))\}^{-1}\varepsilon(s_n,t_n)\varepsilon(s_m,t_m), \]
the inverse taken in the compact group \( H := \varepsilon(S \times \{e\}) \). Choose subnets such that \( \varepsilon(s_{n_\alpha},t_{n_\alpha}) \to a \) and \( \varepsilon(s_{m_\beta},t_{m_\beta}) \to b \) in \( GF \). From (7),
\[ \lim_\beta \lim_\alpha \varepsilon \circ r_T((s_{n_\alpha},t_{n_\alpha})(s_{m_\beta},t_{m_\beta})) \]
\[ = \{\tau_S(ab)\}^{-1}ab \]
\[ = \lim_\beta \lim_\alpha \varepsilon \circ r_T((s_{n_\alpha},t_{n_\alpha})(s_{m_\beta},t_{m_\beta})), \]
the convergence of the term in braces takes place in \( H \) thereby allowing use of the restricted joint continuity property of \( GF \) (see Section 2). Thus, \( \ell_1 = \ell_2 \); hence, \( r^*_T(f) \in WAP(G) \).

\[ \square \]

**Corollary 4.3.** Let \( S \) be a compact topological group, \( T \) a topological semigroup, and let \( F = LCWAP(G) \) or \( F = KWAP(G) \). Suppose that the Zappa product mapping \([·, ·]\) has the property that, for each pair of sequences \( \{s_n\} \) in \( S \) and \( \{t_n\} \) in \( T \), there exist subnets such that
\[ (8) \quad \lim_\beta \lim_\alpha [t_{n_\beta},s_{n_\alpha}] = \lim_\alpha \lim_\beta [t_{n_\beta},s_{n_\alpha}]. \]

Then \( GF \cong S' \times_T T' \) for some compactifications of \( S' \preceq S^{SAP} \) and \( T' \preceq T^{FTP} \). In particular, this holds if \( G \) is a semidirect product with \([·, ·]\) trivial.
Proof. By Theorem 4.2, it suffices to show that \( r_s^*(F) \subseteq F \). We do this only for \( F = \text{LCWAP}(G) \). Let \( f \in F \). From Theorem 4.1, \( r_s^*(f) \in \text{LC}(G) \). To show that \( r_s^*(f) \in \text{WAP}(G) \), let \( \{ (s_m, t_m) \} \) and \( \{ (s_n, t_n) \} \) be sequences in \( G \) such that the limits

\[
\ell_1 = \lim_{m} \lim_{n} f(r_s((s_n, t_n)(s_m, t_m)))
\]

and

\[
\ell_2 = \lim_{n} \lim_{m} f(r_s((s_n, t_n)(s_m, t_m)))
\]

exist. Using the first equality in (6), the compactness of \( S \), and the double limit hypothesis (8), we may choose subnets such that

\[
\lim_{\beta} \lim_{\alpha} \varepsilon \circ r_s((s_n, t_n)(s_{m_\beta}, t_{m_\beta})) = \lim_{\alpha} \lim_{\beta} \varepsilon \circ r_s((s_{n_\alpha}, t_{n_\alpha})(s_{m_\beta}, t_{m_\beta})).
\]

Therefore, \( \ell_1 = \ell_2 \); hence, \( r_s^*(f) \in \text{WAP}(G) \).

Note that, if \( T \) is a topological group, then \( \text{WAP}(G) \subseteq \text{LC}(G) \); hence, Theorem 4.2 and Corollary 4.3 reduce to assertions about weakly almost periodic compactifications.

5. The minimal ideal as a Zappa product.

**Theorem 5.1.** Let \( S \) be a compact topological group, \( T \) a topological semigroup, and \( F \) an \( m \)-admissible and right amenable subalgebra of \( \text{WAP}(G) \) such that \( G^F \cong S' \times_T T' \) for some compactifications \((\theta, S')\) of \( S \) and \((\psi, T')\) of \( T \). Then \( B := q_T^*(F) \) is amenable if and only if \( F \) is amenable and \( M(G^F) \cong S' \times_T M(T') \) (under the restricted Zappa product mappings).

Proof. Assume that \( B \) is amenable so that \( M(T') \) is a compact topological group. Since \( L := S' \times M(T') \) is a left ideal it contains a minimal idempotent \( (a, b) \). Let \( d' := \theta(e) \) be the identity of \( S' \) and \( e' \) the identity of \( M(T') \). From \((a, b)^2 = (a, b)\), we have \( a = a[b, a] \) and \( (b, a)b = b \); hence, \([b, a] = d'\). Also, from (2) and (4),

\[
\langle b, a \rangle = \langle e'b, a \rangle = \langle e', [b, a] \rangle \langle b, a \rangle = \langle e', d' \rangle \langle b, a \rangle = e' \langle b, a \rangle,
\]
which shows that $\langle b, a \rangle \in M(T')$. Since $\langle b, a \rangle b = b$, we see that $b \in M(T')$ and $\langle b, a \rangle = e'$. Thus, for $y \in M(T')$, from (3) and (4)

$$\langle yb, a \rangle = \langle y, [b, a] \rangle \langle b, a \rangle = \langle y, d' e' \rangle = y$$

and

$$[yb, a] = [y, [b, a]] = [y, d'] = d'.$$

Taking $y = b^{-1}$, the inverse of $b$ in $M(T')$, we have $\langle e', a \rangle = b^{-1}$ and $[e', a] = d'$; hence, $(d', e')(a, b) = ([e', a], \langle e', a \rangle b) = (d', e')$. It follows that $(d', e')$ is contained in the minimal left ideal $G^F(a, b)$ and is therefore a minimal idempotent. Since $G^F(d', e') = S' \times T'e' = L$, $L \subseteq M(G^F)$. But, by the right amenability of $F$, $M(G^F)$ is the unique minimal left ideal of $G^F$. Therefore, as sets, $S' \times Z M(T') = M(G^F)$. Since $(x, y) \in M(G^F) \Rightarrow (d', y) \in M(G^F) \Rightarrow ([y, x], \langle y, x \rangle) = (d', y)(x, \psi(e)) \in M(G^F) \Rightarrow \langle y, x \rangle \in M(T')$, we see that $M(G^F) = S' \times_Z M(T')$. Since $S'$ and $M(T')$ are groups, $M(G^F)$ must be a group; hence, $F$ is amenable.

Conversely, suppose that $F$ is amenable and $M(G^F) = S' \times_Z M(T')$. Let $b$ be any idempotent in $M(T')$. Then $(d', b)(d', b) = (d', b^2) = (d', b)$; hence, $(d', b)$ must be the identity of the group $M(G^F)$. Therefore, for any $y \in M(T')$,

$$(d', y) = (d', b)(d', y) = (d', by)$$

and

$$(d', y) = (d', y)(d', b) = (d', yb),$$

which shows that $b$ is an identity for $M(T')$. Thus, $M(T')$ is a group; hence, $B$ is amenable.

\[\square\]

**Corollary 5.2.** Let $S$ and $T$ be topological groups with $S$ compact, and let $F$ be an $m$-admissible subalgebra of $WAP(G)$. Suppose that the Zappa product mapping $[\cdot, \cdot]$ has the double limit property (8). Then $M(G^F) = S' \times_Z M(T')$. 
6. AP Compactifications of $G$.

**Theorem 6.1.** Let $S$ and $T$ be topological semigroups. Then $G^{SAP} \cong S' \times_z T'$ for some topological group compactifications $S'$ of $S$ and $T'$ of $T$ if and only if either of the inclusions (i) $r^*_S(SAP(G)) \subseteq SAP(G)$ or (ii) $r^*_T(SAP(G)) \subseteq SAP(G)$ holds, in which case both hold. In particular, if $G$ is a semidirect product, then $G^{SAP} \cong S' \times_z T'$.

**Proof.** Let $f \in SAP(G)$ and $\widehat{f} \in C(G^{SAP})$ such that $\varepsilon^*(\widehat{f}) = f$, where $\varepsilon := \varepsilon_{SAP}$. The identity $f(s, t) = L(s, e)f(e, t)$ and the relative compactness $L(G)f$ imply that condition (b) of Theorem 3.3 holds. By symmetry, it remains to show that (i) implies (ii). If (i) holds, then $r^*_S$ has an extension $\tau^*_S : G^{SAP} \to G^{SAP}$. The identity

$$(9) \quad \varepsilon(r^*_T(s, t)) = \{\tau^*_S(\varepsilon(s, t))\}^{-1}\varepsilon(s, t)$$

implies that

$$L(s, t)r^*_T(f)(s', t') = \widehat{f}\left(\{\tau^*_S(\varepsilon((s, t), (s', t'))\}\right)^{-1}\varepsilon((s, t), (s', t')),$$

and hence that $r^*_T(f) \in AP(G)$. Therefore, $r^*_T$ has an extension $\tau^*_T : G^{AP} \to G^{SAP}$. If $\varphi : G^{AP} \to G^{SAP}$ denotes the canonical homomorphism, then from (9),

$$\tau^*_T(u) = \{\tau^*_S(\varphi(u))\}^{-1}\varphi(u), \quad u \in G^{AP}.$$ 

It follows easily that $\tau^*_T$ has the distal property

$$\tau^*_T(uvw) = \tau^*_T(uw), \quad u, v^2 = v, \quad w \in G^{AP},$$

which implies that $r^*_T(SAP(G)) \subseteq D(G) \cap AP(G) = SAP(G)$.

In the case of a semidirect product, either $r^*_S$ or $r^*_T$ is a homomorphism; hence, (i) or (ii) holds. \hfill \Box

**Theorem 6.2.** Let $S$ be a compact topological group and $T$ a topological semigroup such that $WAP(G)$ is amenable. If the Zappa product mapping $[\cdot, \cdot]$ has the double limit property (8), then $G^{SAP} \cong S'' \times_z T''$ for some topological group compactifications of $S$ and $T$. In particular, if $G$ is a semidirect product with $[\cdot, \cdot]$ trivial, then $G^{SAP}$ is a semidirect product.
COMPACTIFICATIONS OF ZAPPA PRODUCTS

Proof. Let $F = LCWAP(G)$. By Corollary 4.3, $G^F \cong S' \times T'$ for some compactifications of $S$ and $T$. By Theorem 5.1, $M(G^F) \cong S' \times M(T')$ is a group with identity $(d', e')$; hence, $[e', x] = x$ for all $x \in S'$. Moreover, since $G^F$ is a Zappa product, $r^*_F(F) \subseteq F$; hence, $r_S$ has an extension $\tau_S : G^F \mapsto G^F$.

Now let $f \in F$ and $g = r^*_S(f)$. Set $\varepsilon = \varepsilon_p$, and let $\varepsilon^*(\hat{f}) = f$ and $\varepsilon^*(\hat{g}) = g$, so that $\hat{g} = \tau_S(\hat{f})$. For $(x, y), (u, v) \in G^F$, we have

\[\tau_S((x, y)(d', e')(u, v)) = \tau_S((x, y)((e', u), (e', u)v))\]
\[= (x[y, u], \psi(e))\]
\[= \tau_S((x, y)(u, v));\]

hence, $\hat{g}((x, y)(d', e')(u, v)) = \hat{g}((x, y)(u, v))$. Therefore, $g$ has the distal property; hence, $g \in SAP(G)$. The conclusion now follows from Theorem 6.1. \qed

7. Examples. (a) Let $C$ denote the complex numbers under addition and $C^*$ the nonzero complex numbers under multiplication. Then

\[G = S \times T = (C \times C^*) \times (C \times C^*)\]

is a Zappa product under multiplication

\[(z, a, w, b)(z', a', w', b') = (z + bz', aa', a'w + w', bb').\]

Since $(0, 1, w, b)(z, a, 0, 1) = (bz, a, aw, b)$,

\[[w, b], (z, a)] = (bz, a)\]

and

\[\langle (w, b), (z, a) \rangle = (aw, b) = [(z, a), (w, b)].\]

We show that $G^{AP}$ is isomorphic to the direct product $C^{*AP} \times C^{*AP} = (C^* \times C^*)^{AP}$.

Given a net $\{(s_\alpha, t_\alpha)\}$ in $G$ with $s_\alpha = (z_\alpha, a_\alpha)$ we have, for $t = (w, b),

\[r_S((s, t)(s_\alpha, t_\alpha)) = (s, e)((t, s_\alpha], e)\]
\[= (s, e)(bz_\beta, a_\beta, 0, 1)\]
\[= (s, e)(0, a_\alpha, 0, 1)(0, 1, 0, b)(z_\alpha, 1, 0, 1)(0, 1, 0, b^{-1}).\]
It follows easily that $\varepsilon_{AP}(r_S((s,t)(s_\alpha,t_\alpha)))$ has a subnet that converges in $G^{AP}$ uniformly in $(s,t)$. Therefore, $r^*_T(AP(G)) \subseteq AP(G)$; hence, by Theorem 6.1, $G^{AP} = S^4 \times_Z T^B$ for $A = q^*_S(AP(G))$ and $B = q^*_T(AP(G))$.

Since $AP(G)$ is generated by coefficients $f_{\xi\zeta}(s,t) := (U(s,t)\xi,\zeta)$ of continuous finite dimensional unitary representations $U$ of $G$, $A$ is generated by the coefficients $f_{\xi\zeta}(\cdot,e)$. Since $r^*_S(f_{\xi\zeta}) \in AP(G)$, $U$ has the convergence property that each sequence $\{(s_n,t_n)\}$ in $G$ has a subsequence $\{(s_k,t_k)\}$ such that $U \circ r_S((s_k,t_k)(s,t)) = U(s_k[t_k],s),e$ converges uniformly in $s$. Since $S$ is commutative, the unitary representation $U(\cdot,e)$ is a direct sum of characters, each of which has this convergence property. Therefore, $A$ is generated by the continuous characters $\chi(z,a) = \chi_1(z)\chi_2(a)$ of $S = \mathbb{C} \times \mathbb{C}^*$ with the property that each sequence $(s_n,t_n) = (z_n,a_n,w_n,b_n)$ has a subsequence $(z_k,a_k,w_k,b_k)$ such that $\chi(s_k[t_k],s) = \chi(z_k + b_kz, a_k a)$ converges uniformly in $s = (z,a)$. Taking $a_n = a = 1$ and $b_n$ real, we see that, for some $\theta_1, \theta_2 \in \mathbb{R}$ and all $k$, the sequence

$$\chi(s_k[t_k],s) = \chi_1(z_k + b_kz) = \exp \left[i(\theta_1(x_k + b_kx) + \theta_2(y_k + b_ky))\right]$$

converges uniformly in $x$ and $y \in \mathbb{R}$. This is possible only if $\theta_1 = \theta_2 = 0$. Therefore, the characters $\chi_1$ are trivial and $A$ is generated by the characters $\chi_2$ of $\mathbb{C}^*$ so that $S^4 = \mathbb{C}^{*AP}$. Similarly for $T^B$.

Analogous results hold for the subgroups of $G$ obtained by replacing one or both occurrences of $\mathbb{C}^*$ by the torus $\mathbb{T}$. Also, one may obviously replace $\mathbb{C}$ and $\mathbb{C}^*$ by $\mathbb{R}$ and $\mathbb{R}^*$, respectively. \hfill \Box

**b** Consider, $\mathbb{Z}/q\mathbb{Z}$, the ring of integers mod $q \in \mathbb{N}$. Let $H := (\mathbb{Z}/q\mathbb{Z}, +)$, and let $J$ be a subsemigroup of $(\mathbb{Z}/q\mathbb{Z}, \cdot)$ containing 1. Then

$$G = S \times_Z T := (\mathbb{T} \times H) \times_Z (J \times \mathbb{C})$$

is a Zappa product semigroup under multiplication

$$(a,n,m,z)(a',n',m',z') = (aa',n'm + n,mm',za' + z'),$$

where

$$[(m,z),(a,n)] = (a,mn) \quad \text{and} \quad \langle(m,z),(a,n)\rangle = (m,az).$$
Clearly, $[\cdot,\cdot]$ satisfies the double limit property (8); hence, by Corollary 4.3,

$$(S \times_z T)^{LC \text{WAP}} = (\mathbb{T} \times H)' \times_z (J \times \mathbb{C})'.$$

Moreover, by Theorem 4.1,

$$(S \times_z T)^{LC} = (\mathbb{T} \times H)'' \times_z (J \times \mathbb{C})''.$$  \hfill \Box

(e) Consider $H_4 = \mathbb{Z}^4$ with multiplication

$$(j,k,m,n)(j',k',m',n') = (j + j' + nk' + m'n(n - 1)/2, k + k' + nm', m + m', n + n').$$

Since $(j,k,m,n) = (0,k,m,0)(j,0,0,n) =: st$ and

$$ts = (j,0,0,n)(0,k,m,0) = (0,k+nm,m,0)(j+nk+mn(n-1)/2,0,0,n),$$

$H_4$ is a Zappa product

$$(0,\mathbb{Z},\mathbb{Z},0) \cdot (\mathbb{Z},0,0,\mathbb{Z}) \cong \mathbb{Z}^2 \times_z \mathbb{Z}^2$$

with

$$[t,s] = (0, k + nm, m', 0)$$

and

$$\langle t,s \rangle = (j + nk + mn(n - 1)/2, 0, 0, n).$$

Since $(j,k,m,n) = (j,k,m,0)(0,0,0,n) =: s't'$, $H_4$ is also a semidirect product

$$(\mathbb{Z},\mathbb{Z},\mathbb{Z},0)(0,0,0,\mathbb{Z}) \cong \mathbb{Z}^3 \times_z \mathbb{Z}$$

with $\langle s',t' \rangle$ trivial and

$$[t',s'] = (j + nk + mn(n - 1)/2, k + mn, m, 0).$$

Therefore, $H_4^{\text{AP}} \cong (\mathbb{Z}^3)^A \times_z \mathbb{Z}^{AP}$, where $A$ is generated by the characters $\chi(a,b,c) = \exp (2\pi i (\theta_1 a + \theta_2 b + \theta_3 c))$ of $\mathbb{Z}^3$ with the property that each net

$$\chi(s'_{\alpha}[t'_{\alpha},s'_{\alpha}]) = \exp [2\pi i (\theta_1 a_{\alpha} + \theta_2 b_{\alpha} + \theta_3 c_{\alpha})]$$

where $a_{\alpha} = j_{\alpha} + j + n_{\alpha}k + mn_{\alpha}(n_{\alpha} - 1)/2$, $b_{\alpha} = k_{\alpha} + k + mn_{\alpha}$, and $c_{\alpha} = m_{\alpha} + m$, has a subnet that converges uniformly in $(j,k,m)$. This
clearly forces $\theta_1$ and $\theta_2$ rational, so $H_{4}^{AP}$ is the semidirect product

$$((Z^B)^2 \times Z^{AP}) \times_z Z^{AP},$$

where $B$ is generated by the characters $n \mapsto e^{2\pi i \theta n}$ with $\theta$ rational. It follows as above that $H_{4}^{AP}$ is a Zappa product

$$(Z^B \times Z^{AP}) \times_z (Z^B \times Z^{AP}).$$

\(\square\)

(d) The group $G = T \times Z^3$ with multiplication

$$(\zeta, k, m, n)(\zeta', k', m', n') = (\zeta\zeta' \lambda^{nk'+m'n(n-1)/2}, k + k' + m'n, m + m', n + n'),$$

where $\lambda = e^{2\pi i \theta}$ is fixed with $\theta$ irrational, is a Zappa product

$$(1, Z, Z, 0) \cdot (T, 0, 0, Z) = Z^2 \times_z (T \times Z)$$

and contains a dense isomorphic copy of $H_4$ under $\phi(j, k, m, n) = (\lambda^j, k, m, n)$. Arguing as above, $G^{AP}$ is a Zappa product

$$(Z^B \times Z^{AP}) \times_z (1 \times Z^{AP}) = (Z^B \times Z^{AP}) \times_z Z^{AP},$$

and the induced homomorphism $\phi^{AP} : H_{4}^{AP} \to G^{AP}$ maps the third factor of $(Z^B \times Z^{AP}) \times_z (Z^B \times Z^{AP})$ onto 1 and leaves the other coordinates fixed. \(\square\)

REFERENCES


Department of Mathematics, The George Washington University, Washington, DC, 20052

Email address: hdj@gwu.edu

Department of Mathematics, University of Western Ontario, London, Ontario (Emeritus)

Email address: milnes@uwo.ca