NOTE ON IGUSA’S CUSP FORM OF WEIGHT 35

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ABSTRACT. A congruence relation satisfied by Igusa’s cusp form of weight 35 is presented. As a tool to confirm the congruence relation, a Sturm-type theorem for the case of odd-weight Siegel modular forms of degree 2 is included.

1. Introduction. In [5], Igusa gave a set of generators of the graded ring of degree 2 Siegel modular forms. In these generators, there are four even-weight forms $\varphi_4, \varphi_6, \chi_{10}, \chi_{12}$, and only one odd-weight form, $\chi_{35}$. Here $\varphi_k$ is the normalized Eisenstein series of weight $k$, and $\chi_k$ is a cusp form of weight $k$.

The purpose of this paper is to introduce a strange congruence relation of the odd-weight cusp form $X_{35}$, which is a suitable normalization of $\chi_{35}$ (for the precise definition, see subsection 2.2).

Main result. Denote by $a(T; X_{35})$ the $T$-th Fourier coefficient of the cusp form $X_{35}$. If $T$ satisfies $\det (T) \not\equiv 0 \pmod{23}$, then

$$a(T; X_{35}) \equiv 0 \pmod{23},$$

or equivalently,

$$\Theta(X_{35}) \equiv 0 \pmod{23},$$

where $\Theta$ is the theta operator on Siegel modular forms (for the precise definition, see subsection 2.4).

This result shows that almost all the Fourier coefficients $a(T; X_{35})$ are divisible by 23.
2. Preliminaries.

2.1. Notation. First we confirm the notation. Let $\Gamma_n = Sp_n(\mathbb{Z})$ be the Siegel modular group of degree $n$ and $\mathbb{H}_n$ the Siegel upper-half space of degree $n$. We denote by $M_k(\Gamma_n)$ the $\mathbb{C}$-vector space of all Siegel modular forms of weight $k$ for $\Gamma_n$, and $S_k(\Gamma_n)$ is the subspace of cusp forms.

Any $F(Z)$ in $M_k(\Gamma_n)$ has a Fourier expansion of the form

$$F(Z) = \sum_{T \in L_n} a(T; F) q^T, \quad q^T := e^{2\pi i \text{tr}(TZ)}, \quad Z \in \mathbb{H}_n,$$

where $T$ runs over all elements of $L_n$, and

$$\Lambda_n := \{ T = (t_{ij}) \in \text{Sym}_n(\mathbb{Q}) \mid t_{ii}, 2t_{ij} \in \mathbb{Z} \},$$

$$L_n := \{ T \in \Lambda_n \mid T \text{ is semi-positive definite} \}.$$

In this paper, we deal mainly with the case of $n = 2$. For simplicity, we write

$$T = (m, n, r) \quad \text{for} \quad T = \begin{pmatrix} m & r \\ \frac{r^2}{n} & n \end{pmatrix} \in \Lambda_2.$$

For a subring $R$ of $\mathbb{C}$, let $M_k(\Gamma_n)_R \subset M_k(\Gamma_n)$ denote the $R$-module of all modular forms whose Fourier coefficients lie in $R$.

2.2. Igusa’s generators. Let

$$M(\Gamma_2) = \bigoplus_{k \in \mathbb{Z}} M_k(\Gamma_2)$$

be the graded ring of Siegel modular forms of degree 2. Igusa [5] gave a set of generators of the ring $M(\Gamma_2)$. The set consists of five generators

$$\varphi_4, \quad \varphi_6, \quad \chi_{10}, \quad \chi_{12}, \quad \chi_{35},$$

where $\varphi_k$ is the normalized Eisenstein series on $\Gamma_2$ and $\chi_k$ is a cusp form of weight $k$. Moreover he showed that the even-weight generators $\varphi_4, \varphi_6, \chi_{10}$ and $\chi_{12}$ are algebraically independent. Later, he extended the result to the integral case ([6]). Namely, he gave a minimal set of generators over $\mathbb{Z}$ of the ring

$$M(\Gamma_2)_\mathbb{Z} = \bigoplus_{k \in \mathbb{Z}} M_k(\Gamma_2)_\mathbb{Z}.$$
The set of generators consists of 15 modular forms including the following forms:

\[ X_4 := \varphi_4, \quad X_6 := \varphi_6, \]
\[ X_{10} := -2^{-2}\chi_{10}, \quad X_{12} := 2^2 \cdot 3\chi_{12}, \]
\[ X_{35} := 2^2i\chi_{35}. \]

Of course, these forms have rational integral Fourier coefficients under the following normalization:

\[ a((0,0,0); X_4) = a((0,0,0); X_6) = 1 \]
\[ a((1,1,1); X_{10}) = a((1,1,1); X_{12}) = 1 \]
\[ a((2,3,-1); X_{35}) = 1. \]

2.3. Order and the \(p\)-minimum matrix. We define a lexicographical order “\(>\)” for two different elements \(T = (m,n,r)\) and \(T' = (m',n',r')\) of \(\Lambda_2\) by

\[ T > T' \iff (1) \quad \text{tr}(T) > \text{tr}(T') \]

or

\[ (2) \quad \text{tr}(T) = \text{tr}(T'), \quad m > m' \]

or

\[ (3) \quad \text{tr}(T) = \text{tr}(T'), \quad m = m', \quad r > r'. \]

Let \(p\) be a prime and \(\mathbb{Z}_{(p)}\) the local ring consisting of \(p\)-integral rational numbers. For \(F \in M_k(\Gamma_2)_{\mathbb{Z}_{(p)}}\), we define the \(p\)-minimum matrix \(m_p(F)\) of \(F\) by

\[ m_p(F) := \min\{T \in L_2 \mid a(T; F) \not\equiv 0 \pmod{p}\}, \]

where the “min” is defined in the sense of the above order. If \(F \equiv 0 \pmod{p}\), then we define \(m_p(F) = (\infty)\).
Remark 2.1. The $p$-minimum matrices of Igusa's generators are

\[ m_p(X_4) = m_p(X_6) = (0, 0, 0), \]
\[ m_p(X_{10}) = m_p(X_{12}) = (1, 1, -1), \]
\[ m_p(X_{35}) = (2, 3, -1), \]

for any prime number $p$.

The following properties are essential.

Lemma 2.2. (1) $T_1 \succ T_2$, $S_1 \succ S_2$ implies $T_1 + S_1 \succ T_2 + S_2$.
(2) $T_1 \succ T_2$ implies $T_1 \pm S \succ T_2 \pm S$.
(3) $T + S = T' + S'$, $T \succ T'$ implies $S \prec S'$.
(4) $m_p(F \cdot G) = m_p(F) + m_p(G)$.

Proof. (1), (2) Trivial.

(3) We use (2) without notice. By the assumption $T + S = T' + S'$, we have $T - T' = S' - S$. Then $0_2 \prec T - T' = S' - S$ because of $T \succ T'$. Hence, $S \prec S'$.

(4) Let $m_p(F) = T_0$ and $m_p(G) = T_0'$. Then, for all $T \prec T_0$ (respectively, $T \prec T_0'$), $a(T; F) \equiv 0 \pmod{p}$ and $a(T_0; F) \not\equiv 0 \pmod{p}$ (respectively, $a(T; G) \equiv 0 \pmod{p}$ and $a(T_0; G) \not\equiv 0 \pmod{p}$). Now, recall that the $T$-th Fourier coefficient $a(T; F \cdot G)$ of $F \cdot G$ is given by

\[ a(T; F \cdot G) = \sum_{S, S' \in L_2 \atop S + S' = T} a(S; F)a(S'; G). \]

If $T \prec T_0 + T_0'$, then $T = S + S' \prec T_0 + T_0'$, and hence $S \prec T_0$ or $S' \prec T_0'$ because of (1). In this case, $a(S; F) \equiv 0 \pmod{p}$ or $a(S'; G) \equiv 0 \pmod{p}$. Therefore, $a(S; F)a(S'; G) \equiv 0 \pmod{p}$ for each $S, S'$ with $S + S' \prec T_0 + T_0'$. This implies $a(T; F) \equiv 0 \pmod{p}$ for all $T \prec T_0 + T_0'$.

In order to complete the proof of (4), we need to prove that $a(T_0 + T_0'; F \cdot G) \not\equiv 0 \pmod{p}$. If $S + S' = T_0 + T_0'$, then we have by (3) that $S \prec T_0$, $S' \prec T_0'$ or $S \prec T_0$, $S' \prec T_0'$ or $S = T_0$, $S' = T_0'$. In the first two cases, since $a(S; F) \equiv 0 \pmod{p}$ or $a(S'; G) \equiv 0 \pmod{p}$, we get $a(S; F)a(S'; G) \equiv 0 \pmod{p}$. In the third case,
a(T_0; F)a(T_0'; G) \not\equiv 0 \pmod{p}. \quad \text{Thus, } a(T_0 + T_0'; F \cdot G) \not\equiv 0 \pmod{p},

namely, \( m_p(F \cdot G) = T_0 + T_0' \). This completes the proof of (4). \qed

**Sturm-type theorem.** A Sturm-type theorem for the Siegel modular forms was first given by Poor and Yuen in [7]. Recently Choi, Choie and the first author [4] investigated such a problem in the case of degree 2 and proved some theorems.

We introduce the statement of this theorem for the case of level 1.

**Theorem 2.3** (Choi, Choie and Kikuta [4]). Let \( p \) be a prime with \( p \geq 5 \) and \( k \) an even positive integer. For \( F \in M_k(\Gamma_2)_{\mathbb{Z}(p)} \) with Fourier expansion

\[
F = \sum_{T \in L_2} a(T; F)q^T,
\]

we assume that \( a((m, n, r); F) \equiv 0 \pmod{p} \) for all \( m, n, r \) such that

\( 0 \leq m, n \leq k/10 \) and \( 4mn - r^2 \geq 0 \). Then \( F \equiv 0 \pmod{p} \).

We rewrite this theorem for later use:

**Theorem 2.4.** Let \( p \) be a prime with \( p \geq 5 \). Assume that \( F \in M_k(\Gamma_2)_{\mathbb{Z}(p)} \) satisfies \( m_p(F) \succ ([k/10], [k/10], r_0) \) for the maximum \( r_0 \in \mathbb{Z} \) such that \( ([k/10], [k/10], r_0) \in L_2 \). Then \( m_p(F) = (\infty) \), i.e., \( F \equiv 0 \pmod{p} \).

**Proof.** The assertion follows immediately from the inclusion

(2.1)

\[
\left\{ T \in L_2 \mid T \preceq \left( \left[ \frac{k}{10} \right], \left[ \frac{k}{10} \right], r_0 \right) \right\} \supset \{(m, n, r) \in L_2 \mid m, n \leq \frac{k}{10} \}. \quad \Box
\]

**Remark 2.5.** In general, the converse of inclusion (2.1) is not true. For example, \( ([k/10] + 1, 0, 0) \prec ([k/10], [k/10], r_0) \) (for \( k \geq 20 \)). We need a statement of this type to aid the proof of the next proposition.

In order to prove our main result, we need a Sturm-type theorem for the odd-weight case:
Proposition 2.6. Let $p$ be a prime with $p \geq 5$ and $k$ an odd positive integer. For $F \in M_k(\Gamma_2)_{\mathbb{Z}(p)}$, we assume that
\[ m_p(F) \succ \left( \left\lfloor \frac{k-35}{10} \right\rfloor + 2, \left\lfloor \frac{k-35}{10} \right\rfloor + 3, r_0 - 1 \right), \]
where $r_0 \in \mathbb{Z}$ is the maximum number such that
\[ \left( \left\lfloor \frac{k-35}{10} \right\rfloor, \left\lfloor \frac{k-35}{10} \right\rfloor, r_0 \right) \in L_2. \]
Then $m_p(F) = (\infty)$, namely, $F \equiv 0 \pmod{p}$.

Remark 2.7. When $F \in M_k(\Gamma_2)_{\mathbb{Z}(p)}$ is of odd weight, $X_{35} \cdot F \in M_{k+35}(\Gamma_2)_{\mathbb{Z}(p)}$ is of even weight. Using Theorem 2.3 directly, we have the following statement: If $a((m, n, r); F) \equiv 0 \pmod{p}$ for all $m, n$ and $r$ such that $0 \leq m, n \leq \frac{k+35}{10}$ and $4mn - r^2 \geq 0$, then $F \equiv 0 \pmod{p}$.

For our purposes, however, the estimation of Proposition 2.6 is better than this estimation.

Proof of Proposition 2.6. First note that
\[ M_k(\Gamma_2)_{\mathbb{Z}(p)} = X_{35}M_{k-35}(\Gamma_2)_{\mathbb{Z}(p)} \]
for odd $k$. Hence, there exists $G \in M_{k-35}(\Gamma_2)_{\mathbb{Z}(p)}$ such that $F = X_{35} \cdot G$. Using Lemma 2.2 (4), we get $m_p(F) = m_p(X_{35}) + m_p(G)$. Since $m_p(X_{35}) = (2, 3, -1)$, we have
\[ m_p(G) = m_p(F) - (2, 3, -1) \succ \left( \left\lfloor \frac{k-35}{10} \right\rfloor, \left\lfloor \frac{k-35}{10} \right\rfloor, r_0 \right). \]
It should be noted that Lemma 2.2 (2) is used to get the last inequality. Since $G$ is of even weight, we can apply Theorem 2.4 to $G$. This shows that $F = X_{35} \cdot G \equiv 0 \pmod{p}$. \qed

2.4. Theta operator. In [8], Serre used the theta operator $\theta$ on elliptic modular forms to develop the theory of $p$-adic modular forms:
\[ \theta = q \frac{d}{dq} : f = \sum a(t; f)q^t \longmapsto \theta(f) := \sum t \cdot a(t; f)q^t. \]
Later the operator was generalized to the case of Siegel modular forms:
\[ \Theta : F = \sum a(T; F)q^T \longmapsto \Theta(F) := \sum \det(T) \cdot a(T; F)q^T \]
Moreover, the following fact was proven:

**Theorem 2.8** (Böcherer-Nagaoka [3]). Assume that a prime $p$ satisfies $p \geq n + 3$. Then, for any Siegel modular form $F$ in $M_k(\Gamma_n)_{Z(p)}$, there exists a Siegel cusp form $G$ in $S_{k+p+1}(\Gamma_n)_{Z(p)}$ satisfying

$$\Theta(F) \equiv G \pmod{p}.$$ 

**Example 2.9.** Under the notation in subsection 2.2, we have

$$\Theta(X_6) \equiv 4X_{12} \pmod{5}.$$ 

3. **Main result.** On the basis of the previous preparation, we can now describe our main result.

**Theorem 3.1.** Let $a(T; X_{35})$ denote the Fourier coefficient of $X_{35}$. If $\det(T) \not\equiv 0 \pmod{23}$, then

$$a(T; X_{35}) \equiv 0 \pmod{23},$$

or, equivalently,

$$\Theta(X_{35}) \equiv 0 \pmod{23}.$$ 

**Proof.** Our proof mainly depends on Proposition 2.6 and numerical calculation of the Fourier coefficients of $X_{35}$. If we use the theta operator, this assertion is equivalent to showing that

$$\Theta(X_{35}) \equiv 0 \pmod{23}.$$ 

From Theorem 2.8, there exists a Siegel cusp form $G \in S_{59}(\Gamma_2)_{Z(23)}$ such that

$$\Theta(X_{35}) \equiv G \pmod{23}.$$ 

Therefore, the proof is reduced to showing that

$$G \equiv 0 \pmod{23}.$$ 

We now apply Proposition 2.6 to the form $G$. It then suffices to show that

$$a((m,n,r);G) \equiv 0 \pmod{23} \text{ for } T = (m,n,r)$$

with $\text{tr}(T) = m + n \leq 10.$
Since \( a((m, n, r); G) = -a((n, m, r); G) \) for the odd-weight form \( G \), this statement is equivalent to

\[
a((m, n, r); \Theta(X_{35})) \equiv 0 \pmod{23} \quad \text{for} \ T = (m, n, r)
\]

with \( \text{tr}(T) = m + n \leq 9 \).

We then write down the first part the Fourier expansion of \( X_{35} \) following the order introduced in subsection 2.3. For this, we set

\[
q_{jk} := \exp(2\pi iz_{jk}) \quad \text{for} \quad Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{pmatrix} \in \mathbb{H}_2.
\]

The terms corresponding to \( T = (m, n, r) \) with \( \text{tr}(T) = m + n \leq 9 \) are as follows:

\[
X_{35} = (q_{11}^{-1} - q_{12})q_{11}^2q_{22} + (q_{11}^{-1} + q_{12})q_{11}q_{22}^2
\]
\[
+ (q_{11}^{-3} - 6q_{11}^{-2} + 6q_{12})q_{11}q_{22}^2 + (q_{11}^{-3} + 6q_{12} - 6q_{12} - q_{12}^3)q_{11}q_{22}^2
\]
\[
+ (69q_{11}^{-3} + 2277q_{12}^{-1} - 2277q_{12} - 69q_{12}^3)q_{11}^2q_{22}
\]
\[
+ (q_{11}^{-5} - 32384q_{12}^{-2} - 129421q_{12}^{-1} + 129421q_{12} + 32384q_{12} - q_{12}^5)q_{11}^3q_{22}
\]
\[
+ (-q_{11}^{-5} + 32384q_{12}^{-2} + 129421q_{12}^{-1} - 129421q_{12} - 32384q_{12} + q_{12}^5)q_{11}^3q_{22}
\]
\[
+ (69q_{12}^{-3} - 2277q_{12}^{-1} + 2277q_{12} - 69q_{12}^3)q_{11}^2q_{22}
\]
\[
+ (q_{12}^{-5} - 2277q_{12}^{-1} - 47702q_{12} - 2277q_{12} - 69q_{12}^3)q_{11}^5
\]
\[
+ (32384q_{12}^{-4} - 2184448q_{12}^{-2} - 3203072q_{12}^{-1} + 3203072q_{12} + 2184448q_{12}^{-2} - 32384q_{12}^{-4})q_{11}^3q_{22}
\]
\[
+ (-32384q_{12}^{-4} + 2184448q_{12}^{-2} - 3203072q_{12}^{-1} - 3203072q_{12} + 2184448q_{12}^{-2} + 32384q_{12}^{-4})q_{11}^3q_{22}
\]
\[
+ (q_{12}^{-5} + 2277q_{12}^{-1} + 47702q_{12} - 2277q_{12} + 69q_{12}^3)q_{11}^4q_{22}
\]
\[
+ (-69q_{12}^{-5} + 47702q_{12}^{-3} + 709665q_{12}^{-2} - 709665q_{12} - 47702q_{12} + 69q_{12}^5)q_{11}^2q_{22}
\]
\[
+ (-q_{11}^{-7} + 129421q_{12}^{-1} + 2184448q_{12}^{-4} + 41321984q_{12}^{-4} + 105235626q_{12}^{-1}
\]
\[
- 105235626q_{12}^{-1} + 41321984q_{12}^{-4} - 129421q_{12} + q_{12}^7)q_{11}^3q_{22}
\]
\[
+ (-69q_{12}^{-7} - 32384q_{12}^{-6} + 107121810q_{12}^{-3} - 31380096q_{12}^{-2} + 759797709q_{12}^{-1}
\]
\[
- 759797709q_{12}^{-1} + 31380096q_{12}^{-2} - 107121810q_{12}^{-3} + 32384q_{12}^6 + 69q_{12}^7)q_{11}^2q_{22}
\]
\[
+ (69q_{12}^{-7} + 32384q_{12}^{-6} + 107121810q_{12}^{-3} + 31380096q_{12}^{-2} - 759797709q_{12}^{-1}
\]
\[
+ 759797709q_{12}^{-1} - 31380096q_{12}^{-2} + 107121810q_{12}^{-3} - 32384q_{12}^6 - 69q_{12}^7)q_{11}^2q_{22}
\]
\[
+ (q_{12}^{-7} - 129421q_{12}^{-1} - 2184448q_{12}^{-4} - 41321984q_{12}^{-4} + 105235626q_{12}^{-1}
\]
\[
+ 105235626q_{12}^{-1} + 41321984q_{12}^{-4} + 129421q_{12} - q_{12}^7)q_{11}^2q_{22}
\]
\[
+ (69q_{12}^{-5} - 47702q_{12}^{-3} - 709665q_{12}^{-1} + 709665q_{12} + 47702q_{12} - 69q_{12}^5)q_{11}^2q_{22} + \ldots.
\]
The Fourier coefficients different from \( \pm 1 \) are as follows:

\[
a((4, 1, 2); X_{35}) = -69 = -3 \cdot 23, \quad a((5, 1, 2); X_{35}) = 2277 = 3^2 \cdot 23, \quad a((4, 1, 3); X_{35}) = -1294121 = -17 \cdot 23 \cdot 331, \quad a((4, 2, 3); X_{35}) = -32384 = -2^7 \cdot 17 \cdot 23, \quad a((6, 1, 2); X_{35}) = -47702 = -2 \cdot 17 \cdot 23 \cdot 61, \quad a((5, 1, 3); X_{35}) = -3203072 = -2^{13} \cdot 17 \cdot 23, \quad a((5, 2, 3); X_{35}) = -2184448 = -2^8 \cdot 7 \cdot 23 \cdot 53, \quad a((7, 1, 2); X_{35}) = 709665 = 3 \cdot 5 \cdot 11^2 \cdot 17 \cdot 23, \quad a((6, 1, 3); X_{35}) = 105235626 = 2 \cdot 3 \cdot 23 \cdot 762577, \quad a((6, 2, 3); X_{35}) = 41321984 = 2^9 \cdot 11^2 \cdot 23 \cdot 29, \quad a((5, 1, 4); X_{35}) = 759797709 = 3 \cdot 11 \cdot 23 \cdot 29 \cdot 34519, \quad a((5, 2, 4); X_{35}) = -31380096 = -2^7 \cdot 3 \cdot 11 \cdot 17 \cdot 19 \cdot 23, \quad a((5, 3, 4); X_{35}) = 107121810 = 2 \cdot 3 \cdot 5 \cdot 19 \cdot 23 \cdot 8171.
\]

All of these Fourier coefficients are divisible by 23. On the other hand, if \( a(T; X_{35}) = \pm 1 \) for \( T \) in this range, then \( \det(T) = 23/4 \equiv 0 \pmod{23} \). This fact implies that

\[
a((m, n, r); \Theta(X_{35})) \equiv 0 \pmod{23}
\]

for \( T = (m, n, r) \) with \( \text{tr}(T) = m + n \leq 9 \). Therefore, we obtain

\[
a((m, n, r); G) \equiv 0 \pmod{23}
\]

for \( T = (m, n, r) \) with \( \text{tr}(T) = m + n \leq 9 \). Consequently, we have (3.1). This completes the proof of our theorem. \( \square \)

**Remark 3.2.**

1. The numerical examples of the Fourier coefficients \( a(T; X_{35}) \) in the above are calculated by using Ibukiyama’s determinant expression of \( X_{35} \) (cf. \([1, \text{page 253}]\)).

2. The converse statement of the theorem is not true in general. In fact,

\[
a((1, 6, 1); X_{35}) = 0 \quad \text{and} \quad \det((1, 6, 1)) = 23/4 \equiv 0 \pmod{23}.
\]

3. There are other “modulo 23” congruences for the Siegel modular forms in \([2, \text{Satz 5,(a)}]\). In that case, the congruence is concerned with the Eisenstein lifting of the Ramanujan delta function.

**REFERENCES**


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