ON 2-SG-SEMISIMPLE RINGS

DRISS BENNIS, KUI HU AND FANGGUI WANG

ABSTRACT. In this paper, we investigate 2-SG-semisimple rings which are a particular kind of quasi-Frobenius rings over which all modules are periodic of period 2. Namely, we show that local 2-SG-semisimple rings are the same as the known Artinian valuation rings. Also, a relation between Dedekind domains and 2-SG-semisimple rings is established.

1. Introduction. Throughout this paper, all rings are commutative with identity element and all modules are unital. It is convenient to use $m$-local or (simply) local to refer to not necessarily Noetherian rings with a unique maximal ideal $m$. We assume that the reader is familiar with the Gorenstein homological algebra (some references are [9, 10, 12]).

For a ring $R$ and a positive integer $n \geq 1$, an $R$-module $M$ is said to be $n$-strongly Gorenstein projective ($n$-SG-projective for short), if there exists an exact sequence of $R$-modules

$$0 \rightarrow M \rightarrow P_n \rightarrow \cdots \rightarrow P_1 \rightarrow M \rightarrow 0,$$

where each $P_i$ is projective, such that $\text{Hom}_R(-, Q)$ leaves the sequence exact whenever $Q$ is a projective $R$-module (see [6]). The 1-SG-projective module is simply called strongly Gorenstein projective (SG-projective for short) (see [5]). An extension of these kinds of modules was given in [3]. Namely, we have, for integers $n \geq 1$ and $m \geq 0$, a module $M$ is called $(n, m)$-SG-projective if there exists an exact sequence of modules,

$$0 \rightarrow M \rightarrow Q_n \rightarrow \cdots \rightarrow Q_1 \rightarrow M \rightarrow 0,$$


Keywords and phrases. 2-SG-projective, 2-SG-semisimple rings, quasi-Frobenius rings, factors of Dedekind domains, Artinian valuation rings.

Received by the editors on January 2, 2013, and in revised form on July 23, 2013.

DOI:10.1216/RMJ-2015-45-4-1093 Copyright ©2015 Rocky Mountain Mathematics Consortium

1093
where pd($Q_i$) $\leq m$ for $1 \leq i \leq n$, such that Ext$^i(M, Q) = 0$ for any $i > m$ and for any projective module $Q$. A general study of rings over which every module is $(n, m)$-SG-projective was done in [4], and such rings are called $(n, m)$-SG. Thus, as in classical homological dimension, the $(n, m)$-SG rings with small integers $n$ and $m$ would be of interest. Let us call by $n$-SG-semisimple, for an integer $n \geq 1$, the $(n, 0)$-SG rings. From [4, Corollary 2.8], $n$-SG-semisimple rings are a particular kind of quasi-Frobenius rings. In [8], it was proved that a local ring is 1-SG-semisimple if and only if it contains a unique non-trivial ideal.

The aim of this paper is to study 2-SG-semisimple rings. We prove that 2-SG-semisimple is the same as the well-known Artinian serial rings (see Corollary 2.7). Recall that a ring is called serial if it is a finite direct product of valuation rings, where a ring (not necessarily a domain) is called valuation if the lattice of all its ideals is linearly ordered under inclusions (see, for example, [11, pages 10 and 11]). Namely, we prove that a local ring is 2-SG-semisimple if and only if it is an Artinian valuation ring (see Theorem 2.6). Also, a relation between Dedekind domains and 2-SG-semisimple rings is established in Proposition 2.9.

Before starting, we need to recall some useful results about quasi-Frobenius rings (for more details about these kinds of rings, see, for example, [14]). The quasi-Frobenius rings have several characterizations, and here, we only need the following ones:

**Theorem 1.1** ([14], Theorems 1.50, 7.55 and 7.56). For a ring $R$, the following are equivalent:

(i) $R$ is quasi-Frobenius;
(ii) $R$ is Artinian and self-injective;
(iii) every projective $R$-module is injective;
(iv) every injective $R$-module is projective;
(v) $R$ is Noetherian and, for every ideal $I$, Ann(Ann($I$)) = $I$, where Ann($I$) denotes the annihilator of $I$.

For the local case, we have the following result:

**Theorem 1.2** ([13], Theorems 221). Let $R$ be an $m$-local and zero-dimensional Noetherian ring. The following are equivalent:

(i) $R$ is quasi-Frobenius;
(ii) $\operatorname{Ann}(m)$ is a principal ideal.

We have the following structural characterization of quasi-Frobenius rings.

**Proposition 1.3.** A ring $R$ is quasi-Frobenius if and only if $R = R_1 \times \cdots \times R_n$, where each $R_i$ is a local quasi-Frobenius ring.

**2. Main results.** We aim to give an equivalent characterization of 2-SG-semisimple rings. The following leads us to restrict the study to the case of local rings.

**Lemma 2.1 ([4], Proposition 2.13).** A ring $R$ is 2-SG-semisimple if and only if $R = R_1 \times \cdots \times R_n$, where each $R_i$ is a local 2-SG-semisimple ring.

Before giving the main result, we need the following lemmas.

The following result is a characterization of Artinian valuation local rings.

**Lemma 2.2 ([2], Proposition 8.8).** Let $R$ be an Artinian $m$-local ring. Then the following assertions are equivalent:

(i) every ideal is principal;
(ii) the maximal ideal $m$ is principal;
(iii) $R$ is a valuation ring.

In this case every ideal $I$ of $R$ is of the form $a^nR$ where $a$ generates $m$.

The two results below investigate the 2-SG-projective modules over local quasi-Frobenius rings.

**Lemma 2.3.** Let $R$ be a local quasi-Frobenius ring and $M$ a finitely generated $R$-module. If $M$ is 2-SG-projective, then there is an exact sequence $0 \to M \to F_2 \to F_1 \to M \to 0$ where $F_1$ and $F_2$ are free and
finitely generated $R$-modules. Furthermore, if $M$ is an ideal of $R$, then the exact sequence can be of the form:

$$0 \rightarrow M \rightarrow R \rightarrow R^n \rightarrow M \rightarrow 0,$$

where $n$ is a positive integer.

Proof. Let $M$ be a finitely generated 2-SG-projective $R$-module. Then, by [18, Theorem 3.14], there exists an exact sequence of $R$-modules

$$0 \rightarrow M \rightarrow F_2 \rightarrow F_1 \rightarrow M \rightarrow 0$$

with $F_1$ and $F_2$ are finitely generated projective $R$-modules. Notice that $R$ is local, so $F_1$ and $F_2$ are finitely generated free and the first assertion follows.

Now, suppose that $M$ is an ideal of $R$. Decomposing the exact sequence $0 \rightarrow M \rightarrow F_2 \rightarrow F_1 \rightarrow M \rightarrow 0$ to get the short exact sequences: $0 \rightarrow M \rightarrow F_2 \rightarrow K \rightarrow 0$ and $0 \rightarrow K \rightarrow F_1 \rightarrow M \rightarrow 0$. Since $R$ is quasi-Frobenius, $F_1$ and $R$ are injective $R$-modules. Then, we can apply the dual of the horseshoe lemma [15, Note after Lemma 6.20] to the short exact sequences above with the canonical one, $0 \rightarrow M \rightarrow R \rightarrow R/M \rightarrow 0$, to get the following commutative diagram with exact columns and rows:

$$
\begin{array}{ccc}
0 & 0 & 0 \\
\uparrow & \uparrow & \uparrow \\
0 & R/M & Q & \rightarrow M \rightarrow 0 \\
\uparrow & \uparrow & \uparrow \\
0 & R & R \oplus F_1 & \rightarrow F_1 \rightarrow 0 \\
\uparrow & \uparrow & \uparrow \\
0 & M & F_2 & \rightarrow K \rightarrow 0 \\
\uparrow & \uparrow & \uparrow \\
0 & 0 & 0 \\
\end{array}
$$

From the top horizontal sequence, $Q$ is a Gorenstein projective and finitely generated $R$-module. Then, using the middle vertical sequence, $Q$ has finite projective dimension. This shows, using [12, Proposition 2.27], that $Q$ is projective and then free (since $R$ is local). Then, there is a positive integer $n$ such that $Q \cong R^n$. Finally, combining the top horizontal sequence with the left vertical one to get the desired sequence. \qed
Corollary 2.4. Let $R$ be a local quasi-Frobenius ring, and let $a$ be a zero-divisor element of $R$. If the principal ideal $aR$ is 2-SG-projective, then $\text{Ann} (a)$ is also principal and there are exact sequences of the form:

$$0 \rightarrow aR \rightarrow R \rightarrow R \rightarrow aR \rightarrow 0$$
$$0 \rightarrow \text{Ann} (a) \rightarrow R \rightarrow R \rightarrow \text{Ann} (a) \rightarrow 0$$
$$0 \rightarrow R/aR \rightarrow R \rightarrow R/ \text{Ann} (a) \rightarrow 0$$

Proof. By Lemma 2.3, we have an exact sequence of the form:

$$0 \rightarrow R/aR \rightarrow R^n \rightarrow aR \rightarrow 0$$

where $n$ is a positive integer. By the Schanuel lemma [15, Theorem 9.4 (i)], the above exact sequence with the following canonical one:

$$0 \rightarrow \text{Ann} (aR) \rightarrow R \rightarrow aR \rightarrow 0$$

implies that $\text{Ann} (a) \oplus R^n \cong R/aR \oplus R$. This shows that $\text{Ann} (a)$ must be principal and $n = 1$ which help to construct the desired sequences.

The structure of modules over Artinian serial rings is given by the following well-known result.

Lemma 2.5 ([11], Theorems 5.6). Let $R$ be an Artinian serial ring. Then every $R$-module is a direct sum of cyclic modules.

Now we are in position to give the main result.

Theorem 2.6. An $m$-local ring $R$ is 2-SG-semisimple if and only if it is an Artinian valuation ring.

Proof. If $R$ is 2-SG-semisimple, then it is quasi-Frobenius (by [4, Corollary 2.8]). Then, by Theorem 1.2, $\text{Ann} (m)$ is principal. This shows, using Corollary 2.4 and Theorem 1.1, that $m = \text{Ann} (\text{Ann} (m))$ is principal. Therefore, $R$ is a valuation ring (by Lemma 2.2). Conversely, assume that $R$ is an Artinian valuation ring. Obviously, $R$ is quasi-Frobenius with only principal ideals. Then, for every zero-divisor element $a$ of $R$, we have the exact sequences $0 \rightarrow \text{Ann} (a) \rightarrow R \rightarrow$
\[ aR \to 0 \text{ and } 0 \to aR = \text{Ann}(\text{Ann}(a)) \to R \to \text{Ann}(a) \to 0. \] Combining these sequences, we deduce that \( aR \) is 2-SG-projective. Then, from Corollary 2.4, the cyclic module \( R/aR \) is also 2-SG-projective and so are all cyclic modules including the free ones. Therefore, Lemma 2.5 with [3, Proposition 2.3] show that every module is 2-SG-projective and therefore \( R \) is 2-SG-semisimple. \( \square \)

From Lemma 2.1, the structure of 2-SG-semisimple rings is immediately deduced as follows.

**Corollary 2.7.** A ring \( R \) is 2-SG-semisimple if and only if it is an Artinian serial ring.

To construct examples of 2-SG-semisimple rings, one can use the well-known result that nontrivial factor rings of Dedekind domains are principal Artinian serial rings, which means that nontrivial factor rings of Dedekind domains are 2-SG-semisimple (see, for example, [17, Corollary, page 278]). The following result (Proposition 2.9) shows that the Dedekind domains is closely related to the 2-SG-semisimple rings in the sense that the converse of the well-known result above holds true. To prove this result, we use the following lemma.

**Lemma 2.8.** Let \( R \) be a domain and \( P \) a maximal ideal of \( R \) which is finitely generated. Then \( P \) is invertible if and only if \( P \rangle (\text{i.e., } PR_P) \) is a principal ideal of \( R_P \).

**Proof.** By [16, Theorem 8.4.2], \( P \) is invertible if and only if \( P_m \) is principal for any maximal ideal \( m \) of \( R \). Since \( P \) is maximal, \( P_m = R_m \) for any maximal ideal \( m \) other than \( P \). \( \square \)

**Proposition 2.9.** A domain \( R \) is Dedekind if and only if every nontrivial factor ring of \( R \) is 2-SG-semisimple.

**Proof.** If every nontrivial factor ring of \( R \) is 2-SG-semisimple, then, by [13, Theorem 90], \( R \) must be one-dimensional and Noetherian. So, by [1, Theorem 3], \( R \) must be a Dedekind domain. We give a direct proof here. Let \( P \) be a maximal ideal of \( R \), and let \( a \) be an element in \( P \) which is not in \( P^2 \). Since \( R/P^2 \) is a QF-ring, by Theorem 1.1, \( (\bar{a}) = \text{Ann}(\text{Ann}(\bar{a})) \). Since \((P/P^2)^2 = 0\), it can be seen
that \( \text{Ann}(\text{Ann}(\bar{a})) = P/P^2 \). Therefore \( \bar{a} = P/P^2 \). So \( Ra + P^2 = P \) and by the Nakayama lemma, \( P_P = (a)_P \). Thus, by Lemma 2.8, \( P \) is invertible, and this means that \( R \) is a Dedekind domain.

For the “only if” part, let \( I \) be a proper ideal of a Dedekind domain \( R \). Then \( I = P_1^{t_1}P_2^{t_2} \cdots P_n^{t_n} \) for some prime ideals \( P_1, P_2, \ldots, P_n \) and some integers \( t_1, t_2, \ldots, t_n \). By the Chinese remainder theorem, \( R/I \cong R/P_1^{t_1} \oplus R/P_2^{t_2} \oplus \cdots \oplus R/P_n^{t_n} \). In order to show that \( R/I \) is 2-SG-semisimple, we only need to prove that \( R/P_i^{t_i} \) is such a ring. When \( t_i = 1 \), the field \( R/P_i \) is certainly 2-SG-semisimple. Therefore, we can assume that \( t_i > 1 \). Since \( R/P_i^{t_i} \) is an Artinian local ring, by Lemma 2.2 and Theorem 2.6, it suffices to prove that the maximal ideal \( P_i/P_i^{t_i} \) is principal. By [16, Corollary 9.8.7], we can choose an element \( b \in P_i^{t_i} \) and an element \( c \in P_i \) such that \( P_i = (b, c) \). Therefore, \( P_i/P_i^{t_i} = (c + P_i^{t_i}) \) is principal. \( \Box \)

**Acknowledgments.** The authors would like to thank the referee for careful reading and helpful comments.

**REFERENCES**


Department of Mathematics, Faculty of Sciences, B.P. 1014, Mohammed V University, Rabat, Morocco

Email address: d.bennis@fsr.ac.ma, driss_bennis@hotmail.com

College of Science, Southwest University of Science and Technology, Mianyang, 621010, P.R. China

Email address: hukui200418@163.com

College of Mathematics and Software Science, Sichuan Normal University, Chengdu, 610068, P.R. China

Email address: wangfg2004@163.com