INEQUALITIES FOR SUMS OF INDEPENDENT RANDOM VARIABLES IN LORENTZ SPACES

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ABSTRACT. By using interpolation with a function parameter, we establish a moment inequality for sums of independent random variables in Lorentz spaces $\mathcal{L}^{p,q}(\varphi)$. These estimates generalize Rosenthal inequalities in the Lorentz-Zygmund spaces $\mathcal{L}^{p,q}(\log \mathcal{L})^{q}$ as well as Lorentz spaces $\mathcal{L}^{p,q}$.

1. Introduction. We begin our work by recalling the classical Khintchine inequalities. Let $\{r_k\}_{k \geq 1}$ be a Rademacher sequence on a probability space $(\Omega, \mathcal{F}, P)$. Since $\{r_k\}_{k \geq 1}$ is an orthogonal sequence in $L^2(\Omega)$, for any finite sequence $\{\alpha_k\} \subseteq \mathbb{C}$

$$\left\| \sum_k \alpha_k r_k \right\|_2 = \left( \sum_k |\alpha_k|^2 \right)^{1/2}.$$ 

The classical Khintchine inequalities assert that $\| \sum_k \alpha_k r_k \|_2$ is uniformly equivalent to $\| \sum_k \alpha_k r_k \|_p$ for any $p < \infty$, namely,

$$\left\| \sum_k \alpha_k r_k \right\|_p \approx \left( \sum_k |\alpha_k|^2 \right)^{1/2}.$$ 

The equivalence $A \approx B$ means that $c_1 A \leq B \leq c_2 A$ for some positive constants $c_1$ and $c_2$. Rosenthal [12] generalized the Khintchine inequality by replacing $(r_k)_{k \geq 1}$ with an arbitrary sequence $(X_k)_{k \geq 1}$ of independent symmetric random variables on a probability space $(\Omega, \mathcal{F}, P)$. More precisely, he proved that, for such a sequence $(X_k)_{k \geq 1} \subset \mathcal{L}^p(\Omega)$, $p > 2$, we have

$$\| \sum_{k=1}^n X_k \|_p \approx \max \left\{ \left( \sum_{k=1}^n \|X_k\|^p \right)^{1/p}, \left( \sum_{k=1}^n \|X_k\|_p \right)^{1/p} \right\}$$


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for all $n \geq 1$. Carothers and Dilworth [3] proved an analogous result for some of the Lorentz spaces, namely, for $2 < p < \infty$, $0 < q \leq \infty$, and any independent symmetric random variables $X_1, X_2, \ldots, X_n$,

\begin{equation}
\left\| \sum_{k=1}^{n} X_k \right\|_{L^{p,q}(\Omega)} \approx \max \left\{ \left\| \sum_{k=1}^{n} X_k \right\|_{L^2(\Omega)}, \left\| \sum_{k=1}^{n} \bigoplus X_k \right\|_{L^{p,q}(0,\infty)} \right\},
\end{equation}

where $\sum_{k=1}^{n} \bigoplus X_k$ denotes the disjoint sum of $X_1, X_2, \ldots, X_n$, which is a function on $(0, \infty)$ with $d_X(t) = \sum_{k=1}^{n} d_{X_k}(t)$. For example, we could take $X(t) = \sum_{i=1}^{n} X_i(t-i+1)\chi_{[i-1,i]}$ for $0 \leq t \leq n$. In the setting symmetric function spaces, Johnson and Schechtman [7] established a generalization of Rosenthal inequalities. Recently, Hu [6] generalize Rosenthal inequalities to $p \geq 0$ instead of $p > 2$ and replaced the quantity 2 by $r \in [1, 2]$ for conditionally independent mean zero random variables.

In this paper, by use of interpolation with a function parameter, a moment inequality is proved for sums of independent random variables in Lorentz spaces $\Lambda^q(\Omega)$. These estimates generalize Rosenthal inequalities in the Lorentz-Zygmund spaces $L^{p,q}(\log L)^\gamma$ as well as Lorentz spaces $L^{p,q}$.

2. Lorentz spaces $\Lambda^q(\varphi)$. Let $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite nonatomic measure space. For a given weight $\omega$, let $L^p_\mu(\omega)$ denote the Lebesgue space defined by the norm $\|f\|_{L^p_\mu(\omega)} = \|f\omega\|_{L^p(\mu)}$ and $L^p_\mu(\omega)$ when the measure is $dt/t$ on $\mathbb{R}^+ = (0, \infty)$.

**Definition 2.1.** We say that function $f : (0, \infty) \to (0, \infty)$ belongs to the class $\mathfrak{B}$ if $f(1) = 1$, $f$ is continuous and

$$\overline{f}(t) = \sup_{s>0} \frac{f(ts)}{f(s)} < \infty,$$

for all $0 < t < \infty$.

For such a function $f$, the Boyd upper and lower indices $\alpha_{\overline{f}}$ and $\beta_{\overline{f}}$ ([10]) of $\overline{f}$, which is submultiplicative and Lebesgue-measurable, are
then defined by
\[
\alpha_f = \lim_{t \to +\infty} \frac{\log f(t)}{\log t}, \quad \beta_f = \lim_{t \to 0} \frac{\log f(t)}{\log t}
\]
with
\[-\infty < \beta_f \leq \alpha_f < +\infty.\]

For example, if \(\theta, \gamma \in \mathbb{R}\), then
\[f(t) = t^{\theta}(1 + |\log t|)^{\gamma} \in \mathcal{B}, \quad \overline{f}(t) = t^{\theta}(1 + |\log t|)^{|\gamma|} \text{ and } \alpha_f = \beta_f = \theta.\]

Let \(\varphi \in \mathcal{B}\) and \(0 < q \leq \infty\); the Lorentz space \(\Lambda^q(\varphi)\) is the set of (classical of) \(\mu\)-measurable functions from \(\Omega\) in \(\mathbb{C}\) such that
\[
\|f\|_{\Lambda^q \Omega(\varphi)} := \|f^*\|_{L^q_\varphi(\varphi)} = \left( \int_0^\infty (\varphi(t)f^*(t))^q \frac{dt}{t} \right)^{1/q} < \infty \quad (0 < q < \infty)
\]
\[
\|f\|_{\Lambda^\infty \Omega(\varphi)} := \|f^*\|_{L^\infty_\varphi(\varphi)} = \sup_{t>0} \varphi(t)f^*(t) < \infty,
\]
where \(f^*\) denotes the decreasing rearrangement of \(|f|\), i.e.,
\[
f^*(t) = \inf\{s > 0 : d_f(s) = \mu(\{|f| > s\}) \leq t\}.
\]
It is known that \(\Lambda^q \Omega(\varphi)\) is a rearrangement quasi–Banach space.

**Remark 2.2.** It is also well known that the inclusion relations between Lorentz spaces are determined by their fundamental functions, since \(\Lambda^q_1(\varphi_1) \subset \Lambda^q_1(\varphi_2)\) if and only if \(\omega_2(t) \leq C\omega_1(t)\) for all \(t > 0\), and both spaces agree if and only if \(\omega_1 \approx \omega_2\), where
\[
\omega_i(t) = \left( \int_0^t \varphi(s)^q \frac{ds}{s} \right)^{1/q}
\]
is the fundamental function for \(\Lambda^q_1(\varphi_i), i = 1, 2, [4]\).

**Example 2.3.** For \(\varphi(t) = t^{1/p}(1 + |\log t|)^{\gamma}\) with \(0 < p < \infty\) and \(-\infty < \gamma < +\infty\), \(\Lambda^q_1(\varphi)\) is the Lorentz-Zygmund space \(L^{p,q}(\log L)^{\gamma}\). This is the classical Lorentz space \(L^{p,q}\) if \(\gamma = 0\).

We let \((A_1, A_2)\) denote a compatible couple of quasi-Banach spaces pair (i.e., \(A_1\) and \(A_2\) are quasi-Banach spaces, which are continuously
embedded in some Hausdorff topological vector space) and \( K \) is the classical interpolation functional of Peetre.

\[
K(t, a) = K(t, a, A_1, A_2) = \inf \{ \| a_1 \|_{A_1} + t \| a_2 \|_{A_2} : a = a_1 + a_2 \},
\]

\( t > 0 \).

We can define, for each \( p \), \( 0 < p \leq \infty \) and each Lebesgue-measurable function \( f : (0, \infty) \to (0, \infty) \), the space

\[
(A_1, A_2)_{f,p;K} = \left\{ a : a \in A_1 + A_2, \| a \|_{f,p;K} \leq K(t, a; A_1, A_2)/f(t) \| L^q_{\infty}(0, \infty) \right\}.
\]

The space \((A_1, A_2)_{f,p;K}\) is quasi-normed by \( \| \cdot \|_{f,p;K} \). To generalize to \((A_1, A_2)_{f,p;K}\) the very well known properties of this space when \( f(t) = t^q \) (i.e., \((A_1, A_2)_{\theta,p;K}\)), one takes the function \( f \) in the class \( \mathfrak{B} \).

In [10], Merucci showed that interpolation with a function parameter is perfectly suited for identifying interpolation spaces between two quasi-normed Lorentz spaces \( \Lambda_{\Omega}^q(\varphi) \). We refer the reader to [5, 10, 11, 13] for the theory and bibliography concerning these spaces. Recall also that intersection of two Lorentz spaces \( \Lambda_{\Omega}^q(\varphi_1) \) and \( \Lambda_{\Omega}^q(\varphi_2) \) is a quasi–Banach space under the quasi-norm \( \max\{\| \cdot \|_{\Lambda_{\Omega}^q(\varphi_1)}, \| \cdot \|_{\Lambda_{\Omega}^q(\varphi_2)}\} \).

3. Main results. In the sequel, we assume that \((\Omega, \mathfrak{F}, P)\) is probability space and establish an extension of Rosenthal inequalities in Lorentz spaces \( \Lambda_{\Omega}^q(\varphi) \). To prove the main result, we need the following lemma.

**Lemma 3.1.** Let \( 0 < r < p < \infty \), \( f \in \mathfrak{B} \), and \( 0 < q \leq \infty \). Then

\[
(L^r(0, \infty), L^r(0, \infty) \cap L^p(0, \infty))_{f,q;K} = L^r(0, \infty) \cap (L^r(0, \infty), L^p(0, \infty))_{f,q;K}.
\]

**Proof.** By use of Holmsted’s formula about interpolation with a function parameter \([10, 11]\) the proof of this lemma is similar to [3, Lemma 2.1]. \(\square\)
Theorem 3.2. Given $1 \leq r \leq 2 < p < \infty$ and $0 < q \leq \infty$, let $f \in \mathfrak{B}$ with $0 < \beta_f \leq \alpha_f < 1$. Then

$$\left\| \sum_{k=1}^{n} X_k \right\|_{\Lambda^q_{\Omega}(\varphi)} \approx \max \left\{ \left\| \sum_{k=1}^{n} X_k \right\|_{L^r(\Omega)}, \left\| \sum_{k=1}^{n} \bigoplus X_k \right\|_{\Lambda^q_{(0,\infty)}(\varphi)} \right\},$$

for all independent symmetric random variables $X_1, X_2, \ldots, X_n$ in $\Lambda^q_{\Omega}(\varphi)$, where

$$\varphi(t) = \frac{t^{1/r}}{f(t^{1/r-1/p})}.$$

Proof. It follows from [10, Theorem 3] that $\varphi \in \mathfrak{B}$. It is convenient to take $\Omega$ be $[0,1]^N$ with the product measure and denote a typical element of $\Omega$ by the sequence $t = (t_1, t_2, \ldots)$. Define a linear operator $T : L_{0}(0, \infty) \rightarrow L_{0}(\Omega \times [0,1])$ by

$$T(g) = \sum_{k=1}^{\infty} g_k(t_k) r_k(s),$$

where $g_k(t_k) = g(t_k + k - 1)$ and $r_k(s)$ is the $k$th Rademacher function. Then, by Hu’s inequality [6], $T$ is a bounded operator from $L^r(0, \infty) \cap L^p(0, \infty)$ into $L^p(\Omega \times [0,1])$ for $p > 2$. So, by Lemma 3.1 and the interpolation theorem with a function parameter ([10, Theorem 3] and [5]), $T$ is bounded from $L^r(0, \infty) \cap \Lambda^q_{(0,\infty)}(\varphi)$ into $\Lambda^q_{\Omega}(\varphi)$, where

$$\varphi(t) = \frac{t^{1/r}}{f(t^{1/r-1/p})}.$$

Therefore, there exists a positive constant $C$ such that

$$(3.1) \quad \left\| \sum_{k=1}^{n} X_k \right\|_{\Lambda^q_{\Omega}(\varphi)} \leq C \max \left\{ \left\| \sum_{k=1}^{n} X_k \right\|_{L^r(\Omega)}, \left\| \sum_{k=1}^{n} \bigoplus X_k \right\|_{\Lambda^q_{(0,\infty)}(\varphi)} \right\}.$$

It follows from Remark 2.2 that

$$(3.2) \quad \left\| \sum_{k=1}^{n} X_k \right\|_{L^2(\Omega)} \leq C_{1} \left\| \sum_{k=1}^{n} X_k \right\|_{\Lambda^q_{(\Omega)}(\varphi)}$$

for a positive constant $C_{1}$.
Since \(1 \leq r \leq 2 < p\) and \(\alpha_T < 1\), it follows from [10, Propositions 2, 3] that
\[
\alpha_T(t^{1/r-1/p}) = \left(\frac{1}{r} - \frac{1}{p}\right)\alpha_T,
\]
and so \(\alpha_T < 1\). On the other hand, \(\overline{\alpha}_{\Lambda^q_{\Omega}}(\varphi) = \alpha_T < 1\), where \(\overline{\alpha}_{\Lambda^q_{\Omega}}(\varphi)\) are Boyd indices of \(\Lambda^q_{\Omega}(\varphi)\), [13]. Now, by [8, Theorem 5.8], \(\Lambda^q_{\Omega}(\varphi)\) has the Kalton property (that is, for
\[
\varphi(t) = \frac{t^{1/r}}{f(t^{1/r-1/p})},
\]
\(\Lambda^q_{\Omega}(\varphi)\) satisfies \(\|X\| \leq C\|Y\|\) whenever \(X^{**} \leq Y^{**}\) (recall that \(X^{**}(t) = t^{-1} \int_0^t X^*(s) \, ds\)).

By the definition of the disjoint sum, it is easy to check that
\[
\left(\sum_{k=1}^n \bigoplus X_k\right)^{**} \leq \left(\sum_{k=1}^n X_k^{1/2}\right)^{**}.
\]
Now, by the Kalton property, we have
\[
(3.3) \quad \left\|\sum_{k=1}^n \bigoplus X_k\right\|_{L^q((0,\infty))} \leq C_2 \left\|\sum_{k=1}^n X_k^2\right\|^{1/2}_{\Lambda^q_{\Omega}(\varphi)}
\]
for some positive constant \(C_2\). Since \(\sum_{k=1}^n X_k\) has the same distribution as \(\sum_{k=1}^n X_k(t)r_k(t)\), by the Maurey-Khintchine inequality [9, Theorem 1.d.6] and inequality (3.3), we obtain
\[
(3.4) \quad \left\|\sum_{k=1}^n \bigoplus X_k\right\|_{\Lambda_q^\Omega((0,\infty))} \leq C_3 \left\|\sum_{k=1}^n X_k\right\|_{\Lambda^q_{\Omega}(\varphi)},
\]
for some positive constant \(C_3\). Therefore, by inequalities (3.2) and (3.4), we get
\[
C' \max\left\{\left\|\sum_{k=1}^n X_k\right\|_{L^2(\Omega)}, \left\|\sum_{k=1}^n \bigoplus X_k\right\|_{\Lambda_q^\Omega((0,\infty))}\right\} \leq \left\|\sum_{k=1}^n X_k\right\|_{\Lambda^q_{\Omega}(\varphi)},
\]
where \(C' = 1/\max\{C_1, C_3\}\). So, the desired inequality now follows
easily since $1 \leq r \leq 2$, i.e.,

\begin{equation}
C' \max \left\{ \left\| \sum_{k=1}^{n} X_k \right\|_{L^r(\Omega)}, \left\| \sum_{k=1}^{n} \bigoplus_{x} X_k \right\|_{\Lambda_q(0,\infty)(\varphi)} \right\} \leq \left\| \sum_{k=1}^{n} X_k \right\|_{\Lambda_q^p(\varphi)}.
\end{equation}

Thus, inequalities (3.1) and (3.5) imply that

\[ \left\| \sum_{k=1}^{n} X_k \right\|_{\Lambda_q^p(\varphi)} \approx \max \left\{ \left\| \sum_{k=1}^{n} X_k \right\|_{L^r(\Omega)}, \left\| \sum_{k=1}^{n} \bigoplus_{x} X_k \right\|_{\Lambda_q(0,\infty)(\varphi)} \right\}. \]

**Corollary 3.3.** Given $1 \leq r \leq 2 < p < \infty$ and $0 < q \leq \infty$, we then have

\[ \left\| \sum_{k=1}^{n} X_k \right\|_{L^{p,q}(\log L)^\gamma} \approx \max \left\{ \left\| \sum_{k=1}^{n} X_k \right\|_{L^r(\Omega)}, \left\| \sum_{k=1}^{n} \bigoplus_{x} X_k \right\|_{L^{p,q}(\log L)^\gamma} \right\}, \]

for all independent symmetric random variables $X_1, X_2, \ldots, X_n$ in $L^{p,q}(\log L)^\gamma$.

**Proof.** It is sufficient to consider

\[ f(t) = t^\theta \left( 1 + \frac{pr}{p-r} |\log t| \right)^{-|\gamma|} \]

in Theorem 3.2. \qed

**Remark 3.4.** In the previous corollary, if $\gamma = 0$ and $r = 2$ ($p = q$), then this corollary implies Rosenthal inequalities (1.2) in Lorentz spaces $L^{p,q}$ (spaces $L^p$).

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**REFERENCES**


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