PRIME DIVISORS OF IRREDUCIBLE CHARACTER DEGREES

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ABSTRACT. Let $G$ be a finite group. We denote by $\rho(G)$ the set of primes which divide some character degrees of $G$ and by $\sigma(G)$ the largest number of distinct primes which divide a single character degree of $G$. We show that $|\rho(G)| \leq 2\sigma(G) + 1$ when $G$ is an almost simple group. For arbitrary finite groups $G$, we show that $|\rho(G)| \leq 2\sigma(G) + 1$ provided that $\sigma(G) \leq 2$.

1. Introduction. Throughout this paper, all groups are finite, and all characters are complex characters. The set of all complex irreducible characters of $G$ is denoted by Irr($G$), and we let cd($G$) be the set of all complex irreducible character degrees of $G$. We define $\rho(G)$ to be the set of primes which divide some character degree of $G$. For $\chi \in$ Irr($G$), let $\pi(\chi)$ be the set of all prime divisors of $\chi(1)$, and let $\sigma(\chi) = |\pi(\chi)|$. Moreover, $\sigma(G)$ is defined to be the maximum value of $\sigma(\chi)$ when $\chi$ runs over the set Irr($G$). Huppert’s $\rho - \sigma$ conjecture proposed by Huppert in [7] states that if $G$ is a solvable group, then $|\rho(G)| \leq 2\sigma(G)$; and, if $G$ is an arbitrary group, then $|\rho(G)| \leq 3\sigma(G)$. For solvable groups, this conjecture has been verified by Manz [11] and Gluck [6] when $\sigma(G) = 1$ and 2, respectively. In general, it is proved by Manz and Wolf [13] that $|\rho(G)| \leq 3\sigma(G) + 2$. For arbitrary groups, Manz [12] showed that $|\rho(G)| = 3$ if $G$ is nonsolvable and $\sigma(G) = 1$. Recently, it has been proved by Casolo and Dolfi [3] that $|\rho(G)| \leq 7\sigma(G)$ for any arbitrary groups $G$. In [13], Manz and Wolf proposed that, for any group $G$,

$$|\rho(G)| \leq 2\sigma(G) + 1.$$  

We call this new conjecture the strengthened Huppert’s $\rho - \sigma$ conjecture. Obviously, this new conjecture is stronger than the original one. In


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this paper, we first improve the result due to Alvis and Barry in [1] by proving the following.

**Theorem A.** Let $G$ be an almost simple group. Then $|\rho(G)| \leq 2\sigma(G)$ unless $G \cong \text{PSL}_2(2^f)$ with $f \geq 2$ and $|\pi(2^f - 1)| = |\pi(2^f + 1)|$. For the exceptions, we have $|\rho(G)| = 2\sigma(G) + 1$.

This verifies the strengthened Huppert’s $\rho - \sigma$ conjecture for almost simple groups. In the next theorem, we verify this new conjecture for groups $G$ with $\sigma(G) \leq 2$.

**Theorem B.** Let $G$ be a finite group. If $\sigma(G) \leq 2$, then $|\rho(G)| \leq 2\sigma(G) + 1$.

Notice that Theorem B is also a generalization to [19, Theorem A].

**Notation.** For a positive integer $n$, we denote the set of all prime divisors of $n$ by $\pi(n)$. If $G$ is a group, then we write $\pi(G)$ instead of $\pi(|G|)$ for the set of all prime divisors of the order of $G$. If $N \trianglelefteq G$ and $\theta \in \text{Irr}(N)$, then the inertia group of $\theta$ in $G$ is denoted by $I_G(\theta)$. We write $\text{Irr}(G|\theta)$ for the set of all irreducible constituents of $\theta^G$. Moreover, if $\chi \in \text{Irr}(G)$, then $\text{Irr}(\chi|N)$ is the set of all irreducible constituents of $\chi$ when restricted to $N$. Recall that a group $G$ is said to be an almost simple group with socle $S$ if there exists a nonabelian simple group $S$ such that $S \trianglelefteq G \leq \text{Aut}(S)$. The greatest common divisor of two integers $a$ and $b$ is $\text{gcd}(a,b)$. Denote by $\Phi_k := \Phi_k(q)$ the value of the $k$th cyclotomic polynomial evaluated at $q$. Other notation is standard.

2. **Proof of Theorem A.** If $G$ is an almost simple group, then $G$ has no normal abelian Sylow subgroup and so, by Ito-Michler’s theorem [14, Theorem 5.4], $\rho(G) = \pi(G)$. This fact will be used without any further reference.

**Lemma 2.1.** Let $S$ be a sporadic simple group, the Tits group or an alternating group of degree at least 7. If $G$ is an almost simple group with socle $S$, then

$$|\pi(G)| = |\pi(S)| \leq 2\sigma(G).$$
Proof. Observe first that, if $S$ is one of the simple groups in the lemma, and $G$ is any almost simple group with socle $S$, then $\pi(G) = \pi(S)$. Since $S \leq G$, we see that $\sigma(S) \leq \sigma(G)$. Thus, it suffices to show that $|\pi(S)| \leq 2\sigma(S)$. By using [4], we can easily check that $|\pi(S)| \leq 2\sigma(S)$ when $S$ is a sporadic simple group, the Tits group or an alternating group of degree $n$ with $7 \leq n \leq 14$. Finally, if $S \cong A_n$ with $n \geq 15$, then the result in [2] yields that $|\pi(S)| = \sigma(S)$. This completes the proof. □

For $\epsilon = \pm$, we use the convention that $\text{PSL}_n^\epsilon(q)$ is $\text{PSL}_n(q)$ if $\epsilon = +$ and $\text{PSU}_n(q)$ if $\epsilon = -$. Let $q \geq 2$ and $n \geq 3$ be integers with $(n, q) \neq (6, 2)$. A prime $\ell$ is called a primitive prime divisor of $q^n - 1$ if $\ell \mid q^n - 1$ but $\ell \nmid q^m - 1$ for any $m < n$. By Zsigmondy’s theorem [21], the primitive prime divisors of $q^n - 1$ always exist. We denote by $\ell_n(q)$ the smallest primitive prime divisor of $q^n - 1$. In Table 1, which is taken from [10], we give the orders of two maximal tori $T_i$ and the corresponding two primitive prime divisors $\ell_i$, for $i = 1, 2$, of classical groups.

Let $C$ be the set consisting of the following simple groups:

\begin{align*}
\text{PSL}_2(q), & \quad \text{PSL}_3(q), \quad \text{PSU}_3(q), \quad \text{PSp}_4(q) \quad \text{PSL}_4(2), \\
\text{PSL}_6(2), & \quad \text{PSL}_7(2), \quad \text{PSU}_4(2), \quad \text{PSU}_4(3), \quad \text{PSU}_6(2), \\
\text{Sp}_4(2)' , & \quad \text{Sp}_6(2), \quad \text{Sp}_8(2), \quad \Omega_7(3), \quad \Omega_8^+(2), \\
\Omega_8^-(2), & \quad 3\text{D}_4(2), \quad G_2(2)', \quad G_2(3), \quad G_2(4).
\end{align*}

**Lemma 2.2.** Let $S$ be a finite simple group of Lie type in characteristic $p$ which is not the Tits groups nor $\text{PSL}_2(2^f)$ with $f \geq 2$. Then $|\pi(S)| \leq 2\sigma(S)$.

Proof. We consider the following cases.

**Case 1.** $S \cong \text{PSL}_2(q)$, where $q = p^f \geq 5$ is odd.

Since $\text{PSL}_2(5) \cong \text{PSL}_2(4)$, we can assume that $q > 5$. In this case, all character degrees of $S$ divide $q, q - 1$ or $q + 1$. Observe that

$\pi(S) = \{p\} \cup \pi(q - 1) \cup \pi(q + 1), \{p\} \cap \pi(q \pm 1) = \emptyset$

and

$\pi(q - 1) \cap \pi(q + 1) = \{2\}$. 

Table 1. Two tori for classical groups.

| $G = G(q)$ | $|T_1|$ | $|T_2|$ | $\ell_1$ | $\ell_2$ |
|---|---|---|---|---|
| $A_n$ | $(q^{n+1}-1)/ (q-1)$ | $q^n-1$ | $\ell_{n+1}(q)$ | $\ell_n(q)$ |
| $^2A_n$, $n \equiv 0(4)$ | $(q^{n+1}+1)/(q+1)$ | $q^n-1$ | $\ell_{2n+2}(q)$ | $\ell_n(q)$ |
| $^2A_n$, $n \equiv 1(4)$ | $(q^{n+1}-1)/(q+1)$ | $q^n+1$ | $\ell_{(n+1)/2}(q)$ | $\ell_{2n}(q)$ |
| $^2A_n$, $n \equiv 2(4)$ | $(q^{n+1}+1)/(q+1)$ | $q^n-1$ | $\ell_{2n+2}(q)$ | $\ell_{n/2}(q)$ |
| $^2A_n$, $n \equiv 3(4)$ | $(q^{n+1}-1)/(q+1)$ | $q^n+1$ | $\ell_{n+1}(q)$ | $\ell_{2n}(q)$ |
| $B_n, C_n$, $n \geq 3$ odd | $q^n+1$ | $q^n-1$ | $\ell_{2n}(q)$ | $\ell_n(q)$ |
| $B_n, C_n$, $n \geq 2$ even | $q^n+1$ | $(q^{n-1}+1)(q+1)$ | $\ell_{2n}(q)$ | $\ell_{2n-2}(q)$ |
| $D_n$, $n \geq 5$ odd | $(q^{n-1}+1)(q+1)$ | $q^n-1$ | $\ell_{2n-2}(q)$ | $\ell_n(q)$ |
| $D_n$, $n \geq 4$ even | $(q^{n-1}+1)(q+1)$ | $(q^{n-1}-1)(q-1)$ | $\ell_{2n-2}(q)$ | $\ell_{n-1}(q)$ |
| $^2D_n$ | $q^n+1$ | $(q^{n-1}+1)(q-1)$ | $\ell_{2n}(q)$ | $\ell_{2n-2}(q)$ |

Hence, we obtain that

$$|\pi(S)| = 1 + \sigma(q+1) + \sigma(q-1) - |\pi(q-1) \cap \pi(q+1)|$$

$$= \sigma(q+1) + \sigma(q-1) \leq 2\sigma(S).$$

**Case 2.** $S \cong \text{PSL}^\epsilon_3(q)$ with $q = p^f$ and $\epsilon = \pm$. As $\text{PSL}_3(2) \cong \text{PSL}_2(7)$ and $\text{PSU}_3(2)$ are not simple, we can assume that $q > 2$. The cases when $q = 3$ or $4$ can be checked directly using [4]. So, we can assume that $q \geq 5$. By [17], $S$ possesses irreducible characters $\chi_i$, $i = 1, 2$, with degree

$$\chi_1(1) = (q-\epsilon 1)^2(q+\epsilon 1) \quad \text{and} \quad \chi_2(1) = q(q^2 + \epsilon q + 1).$$
Let $d = \gcd(3, q - \epsilon 1)$. Then

$$|S| = \frac{1}{d}q^3(q^2 - 1)(q^3 - \epsilon 1) = \frac{1}{d}q^3(q - \epsilon 1)^2(q + \epsilon 1)(q^2 + \epsilon q + 1),$$

and so

$$\pi(S) = \pi(\chi_1) \cup \pi(\chi_2).$$

Therefore, $|\pi(S)| \leq 2\sigma(S)$ as wanted.

**Case 3.** $S \cong \text{PSp}_4(q)$ with $q = p^f > 2$.

By [5, 18], $S$ has two irreducible characters $\chi_i$, $i = 1, 2$, with degrees $\Phi_1^2 \Phi_2^2$ and $q \Phi_1 \Phi_4$, respectively. Since

$$|S| = \frac{1}{d}q^4 \Phi_1^2 \Phi_2^2 \Phi_4$$

where $d = \gcd(2, q - 1)$, we deduce that

$$\pi(S) = \pi(\chi_1) \cup \pi(\chi_2),$$

and thus $|\pi(S)| \leq 2\sigma(S)$.

**Case 4.** $S$ is one of the remaining simple groups in the list $C$.

Using [4], it is routine to check that $|\pi(S)| \leq 2\sigma(S)$ in all these cases.

**Case 5.** $S$ is not in the list $C$.

We consider the following setup. Let $G$ be a simple, simply connected algebraic group defined over a field of size $q$ in characteristic $p$, and let $F$ be a Frobenius map on $G$ such that $S \cong L/Z$, where $L := GF$ and $Z := Z(L)$. Let the pair $(G^*, F^*)$ be dual to $(G, F)$ and let $L^* := G^*F^*$. By Lusztig’s theory, the irreducible characters of $GF$ are partitioned into rational series $E(G^*, (s))$ which are indexed by $(G^*F^*)$-conjugacy classes $(s)$ of semisimple elements $s \in G^*F^*$. Furthermore, if $\gcd(|s|, |Z|) = 1$, then every $\chi \in E(G^*, (s_i))$ is trivial at $Z$, and thus $\chi \in \text{Irr}(S) = \text{Irr}(L/Z)$. (See [15, page 349]). Notice also that $\chi(1)$ is divisible by $|L^* : C_{L^*}(s)|_{p'}$.

For simple classical groups of Lie type, the restriction on $S$ guarantees that both primitive prime divisors $\ell_i$ in Table 1 exist. Let $s_i \in G^*F^*$ with $|s_i| = \ell_i$, $i = 1, 2$. Then $C_{L^*}(s_i) = T_i$ for $i = 1, 2,$
where $T_i$ are maximal tori of $L^*$. Similarly, for each simple exceptional group of Lie type $S$, by \cite[Lemma 2.3]{15}, one can find two semisimple elements $s_i \in G_{F^r}$ with $|s_i| = \ell_i$, $i = 1, 2$. In both cases, we have that $(\ell_i, |Z|) = 1$ for $i = 1, 2$ and, if $a := \gcd(|C_{L^*}(s_1)|, |C_{L^*}(s_2)|)$, then either $a = 1$ or, if a prime $r$ divides $a$, then $r$ also divides $|L^* : C_{L^*}(s_i)|_{p'}$ for some $i$. Let $\chi_i \in \mathcal{E}(G, (s_i))$, $i = 1, 2$, be such that $\chi_i(1) = |L^* : C_{L^*}(s_i)|_{r'}$. Then $\chi_i \in \text{Irr}(S)$ for $i = 1, 2$ and

$$\pi(S) = \{p\} \cup \pi(\chi_1) \cup \pi(\chi_2).$$

Notice that $p$ is relatively prime to both $\chi_i(1)$ for $i = 1, 2$. So,

$$|\pi(S)| = |\{p\} \cup \pi(\chi_1) \cup \pi(\chi_2)|$$

$$= 1 + |\pi(\chi_1)| + |\pi(\chi_2)| - |\pi(\chi_1) \cap \pi(\chi_2)|$$

$$= \sigma(\chi_1) + \sigma(\chi_2) - (|\pi(\chi_1) \cap \pi(\chi_2)| - 1)$$

$$\leq 2\sigma(S) - (|\pi(\chi_1) \cap \pi(\chi_2)| - 1).$$

If we can show that $|\pi(\chi_1) \cap \pi(\chi_2)| \geq 1$, then clearly $|\pi(S)| \leq 2\sigma(S)$, and we are done. By way of contradiction, assume that $\pi(\chi_1) \cap \pi(\chi_2)$ is empty. Then $\gcd(\chi_1(1), \chi_2(1)) = 1$, and so

$$\gcd(|L^* : C_{L^*}(s_1)|_{p'}, |L^* : C_{L^*}(s_2)|_{p'}) = 1.$$ 

It follows that $|L^*|_{p'}$ must divide $|C_{L^*}(s_1)|_{p'} \cdot |C_{L^*}(s_2)|_{p'}$. However, we can check by using \cite[Lemma 2.3]{15} and Table 1 that this is not the case. The proof is now complete. 

We now prove Theorem A which we restate here.

**Theorem 2.3.** Let $G$ be an almost simple group. Then $|\rho(G)| \leq 2\sigma(G)$ unless $G \cong \text{PSL}_2(2^f)$ with $|\pi(2^f - 1)| = |\pi(2^f + 1)|$. For the exceptions, we have $|\rho(G)| = 2\sigma(G) + 1$.

**Proof.** Let $G$ be an almost simple group with simple socle $S$. Since $S \leq G$, we obtain that $\sigma(S) \leq \sigma(G)$.

**Case 1.** $S \cong \text{PSL}_2(q)$ with $q = 2^f \geq 4$.

It is well known that $|S| = q(q^2 - 1)$, $\gcd(2^f - 1, 2^f + 1) = 1$ and

$$\text{cd}(S) = \{1, q - 1, q, q + 1\}.$$
If $|\pi(q - 1)| = |\pi(q + 1)|$, then

$$\pi(S) = \{2\} \cup \pi(q - 1) \cup \pi(q + 1),$$

and thus $|\pi(S)| = 2\sigma(S) + 1$ as $\sigma(S) = |\pi(2^f \pm 1)|$. Assume that $|\pi(q - 1)| \neq |\pi(q + 1)|$. Then $|\pi(2^f + \delta)| > |\pi(2^f - \delta)|$ for some $\delta \in \{\pm 1\}$. Hence, $\sigma(S) = |\pi(2^f + \delta)|$, and thus

$$|\pi(S)| = |\{2\} \cup \pi(2^f - \delta) \cup \pi(2^f + \delta)|$$

$$= 1 + |\pi(2^f - \delta)| + |\pi(2^f + \delta)|.$$

Since $|\pi(2^f + \delta)| \geq |\pi(2^f - \delta)| + 1$, we obtain that

$$|\rho(S)| \leq 2|\pi(2^f + \delta)| \leq 2\sigma(S).$$

Thus, the result holds when $G = S$.

Assume now that $|G : S|$ is nontrivial. We know that $\text{Aut}(S) = S \cdot \langle \varphi \rangle$, where $\varphi$ is a field automorphism of $S$ of order $f$. Thus, $G = S \cdot \langle \psi \rangle$, with $\psi \in \langle \varphi \rangle$. If $f = 2$, then $G \cong A_5 \cdot 2$, and obviously $|\pi(G)| \leq 2\sigma(G)$. Hence, we can assume that $f > 2$. Clearly, if $f \equiv 2 \pmod{4}$ and $G = S \cdot \langle \varphi \rangle$, then $|G : S| > 2$. So by [20, Theorem A], $G$ has two irreducible characters $\chi_i \in \text{Irr}(G)$, $i = 1, 2$, with $\chi_1(1) = |G : S|(q - 1)$ and $\chi_2(1) = |G : S|(q + 1)$. Obviously,

$$\pi(G) = \{2\} \cup \pi(\chi_1) \cup \pi(\chi_2)$$

and

$$\pi(\chi_1) \cap \pi(\chi_2) = \pi(|G : S|) \neq \emptyset.$$

If $|G : S|$ is even, then

$$|\rho(G)| = |\pi(\chi_1) \cup \pi(\chi_2)| \leq |\pi(\chi_1)| + |\pi(\chi_2)| \leq 2\sigma(G).$$

If $|G : S|$ is odd, then

$$|\rho(G)| = |\{2\} \cup \pi(\chi_1) \cup \pi(\chi_2)|$$

$$= 1 + |\pi(\chi_1)| + |\pi(\chi_2)| - |\pi(\chi_1) \cap \pi(\chi_2)|$$

$$= \sigma(\chi_1) + \sigma(\chi_2) - (|\pi(|G : S|)| - 1)$$

$$\leq \sigma(\chi_1) + \sigma(\chi_2)$$

$$\leq 2\sigma(G).$$
Case 2. $S$ is a sporadic simple group, the Tits group or an alternating group of degree at least 7.

By Lemma 2.1, we obtain that $|\rho(G)| \leq 2\sigma(G)$.

Case 3. $S$ is a finite simple group of Lie type in characteristic $p$ and $S$ is not the Tits group nor $\text{PSL}_2(2^f)$ with $f \geq 2$.

Subcase 3a. $\pi(G) = \pi(S)$.

By Lemma 2.2, we have that $|\pi(S)| \leq 2\sigma(S)$. Thus,

$$|\rho(G)| = |\pi(S)| \leq 2\sigma(S) \leq 2\sigma(G).$$

Subcase 3b. $\pi := \pi(G) - \pi(S)$ is nonempty.

Let $A$ be the subgroup of the group of coprime outer automorphisms of $S$ induced by the action of $G$ on $S$. By [15, Lemma 2.10], $A$ is cyclic and central in $\text{Out}(S)$. Moreover, $A$ is generated by a fixed field automorphism $\gamma \in \text{Out}(S)$. It follows that the group $S \cdot A$ is normal in $G$ and $\pi(S \cdot A) = \pi(G)$. Thus we can assume that $G = S \cdot A$ with $A = \langle \gamma \rangle$ and $\gamma$ a field automorphism of $S$. Furthermore, $\pi(\gamma) = \pi$.

Replacing $A$ by a normal subgroup if necessary, we can also assume that $|A| = |\gamma|$ is the product of all distinct primes in $\pi$.

As in the proof of Lemma 2.2, let $\mathcal{G}$ be a simple, simply connected algebraic group defined over a field of size $q = p^f$ in characteristic $p$, and let $F$ be a Frobenius map of $\mathcal{G}$ such that $S \cong L/Z$, where $L := \mathcal{G}^F$ and $Z := Z(L)$. Let the pair $(\mathcal{G}^*, F^*)$ be dual to $(\mathcal{G}, F)$, and let $L^* := \mathcal{G}^{*F^*}$. As $\pi \subseteq \pi(f)$, where $\pi = \pi(G) - \pi(S)$, it is easy to check that both the primitive prime divisors in [15, Lemmas 2.3 and 2.4] exist, and thus one can find two semisimple elements $s_i \in \mathcal{G}^{*F^*}$ with $|s_i| = \ell^i$ such that $(\ell^i, |Z|) = 1$ for $i = 1, 2$. Arguing as in the proof of Lemma 2.2, we obtain that

$$\pi(S) = \{p\} \cup \pi(\chi_1) \cup \pi(\chi_2),$$

where $\chi_i \in \mathcal{G}^{\mathcal{G}^F, (s_i)}$ such that $\chi_i(1) = |L^*: C_{L^*}(s_i)|_p$ and $\chi_i$ can be considered as characters of $S$ for $i = 1, 2$.

We next claim that the inertia group for both $\chi_i$, $i = 1, 2$, in $G$ is exactly $S$. It suffices to show that no field automorphism of $S$ of
prime order can fix $\chi_i$ for $i = 1, 2$. Let $\tau$ be a field automorphism of $S$ of prime order $s$. We can extend $\tau$ to an automorphism of $G^F$ and $G^{*F^*}$, which we also denote by $\tau$. Notice that $C_{G^{*F^*}}(\tau)$ is a finite group of Lie type of the same type as that of $G^{*F^*}$ but defined over a field of size $q^{1/s}$. Now it is straightforward to check that both $\ell_i$, $i = 1, 2$, are relatively prime to $|C_{G^{*F^*}}(\tau)|$. Hence, $G^{*F^*}$-conjugacy classes $(s_i)$ of $s_i$ in $G^{*F^*}$ are not $\tau$-invariant for $i = 1, 2$ (see [15, Proposition 2.6]). Then $\tau(s_i)$ and $s_i$ are not $G^{*F^*}$-conjugate for $i = 1, 2$, and thus $\chi_i \in \mathcal{E}(G^F, (s_i))$, $i = 1, 2$, are not $\tau$-invariant (see [15, Theorem 2.7]). Therefore, we obtain that $\chi_i^G \in \text{Irr}(G)$ for $i = 1, 2$; hence, $\chi_i^G(1) \in \text{cd}(G)$. Since

$$\pi(S) = \{p\} \cup \pi(\chi_1) \cup \pi(\chi_2)$$

and $\pi(G) = \pi(S) \cup \pi([G : S])$, we obtain that

$$\pi(G) = \{p\} \cup \pi([G : S]|\chi_1(1)) \cup \pi([G : S]|\chi_2(1)) = \{p\} \cup \pi(\chi_1^G) \cup \pi(\chi_2^G).$$

Moreover, $p \not| [G : S]|\chi_i(1) = \chi_i^G(1)$ for $i = 1, 2$, and

$$|\pi(\chi_1^G) \cap \pi(\chi_2^G)| \geq 1.$$

Therefore,

$$|\pi(G)| = 1 + \sigma(\chi_1^G) + \sigma(\chi_2^G) - |\pi(\chi_1^G) \cap \pi(\chi_2^G)|$$

$$\leq 2\sigma(G) - (|\pi(\chi_1^G) \cap \pi(\chi_2^G)| - 1)$$

$$\leq 2\sigma(G).$$

The proof is now complete. \qed

The next results will be needed in the proof of Theorem B.

**Lemma 2.4.** Let $S$ be a nonabelian simple group. If $\sigma(S) \leq 2$, then $S$ is one of the following groups.

(i) $S \cong \text{PSL}_2(2^f)$ with $|\pi(2^f \pm 1)| \leq 2$, and so $|\pi(S)| \leq 5$.

(ii) $S \cong \text{PSL}_2(q)$ with $q > 5$ odd and $|\pi(q \pm 1)| \leq 2$ and so $|\pi(S)| \leq 4$.

(iii) $S \in \{M_{11}, A_7, 3B_2(8), 2B_2(32), \text{PSL}_3^+(3), \text{PSL}_3^+(4), \text{PSL}_3(8)\}$ and $|\pi(S)| = 4$. 

Proof. As $S$ is a nonabelian simple group, we have that $|\pi(S)| \geq 3$. If $S \cong \text{PSL}_2(q)$ with $q \geq 4$, then the lemma follows easily as the character degree set of $S$ is known. Now assume that $S \not\cong \text{PSL}_2(q)$. Then Lemmas 2.2 and 2.1 imply that $|\pi(S)| \leq 2\sigma(S)$. So, $3 \leq |\pi(S)| \leq 4$. By checking the list of nonabelian simple groups with at most four prime divisors in [8], we deduce that only those nonabelian simple groups appearing in (iii) above satisfy the hypotheses of the lemma. \qed

**Lemma 2.5.** Let $G$ be an almost simple group with simple socle $S$. If $\sigma(G) \leq 2$, then $\pi(G) = \pi(S)$, where $S$ is one of the simple groups in Lemma 2.4.

**Proof.** Since $\sigma(S) \leq \sigma(G) \leq 2$, we deduce that $S$ is isomorphic to one of the simple groups in the conclusion of Lemma 2.4. If $|\pi(S)| = 3$, then $S$ is one of the simple groups in [8, Table 1], and we can check that $\pi(G) = \pi(S)$ in these cases. Thus, we assume that $|\pi(S)| \geq 4$. Now, if $G = S$, then we have nothing to prove. So, we assume that $G \neq S$. In particular, $G \not\cong \text{PSL}_2(2^f)$ with $f \geq 2$. Then $|\pi(G)| \leq 2\sigma(G) \leq 4$ by Theorem A, and thus $4 \leq |\pi(S)| \leq |\pi(G)| \leq 4$, which forces $|\pi(S)| = |\pi(G)|$ and, hence, $\pi(G) = \pi(S)$ as wanted. \qed

3. **Proof of Theorem B.** The following two lemmas are obvious.

**Lemma 3.1.** Let $A$ and $B$ be groups such that $|\rho(A)| \geq 3$ and $|\rho(B)| \geq 3$. If

$$\sigma(A \times B) \leq 2,$$

then $\sigma(A) = 1 = \sigma(B)$.

**Lemma 3.2.** Let $N$ be a normal subgroup of a group $G$. If $\rho(G/N) = \pi(G/N)$, then

$$\rho(G) - \rho(G/N) \subseteq \rho(N).$$

Recall that the solvable radical of a group $G$ is the largest normal solvable subgroup of $G$.

**Lemma 3.3.** Let $G$ be a nonsolvable group, and let $N$ be the solvable radical of $G$. Suppose that $\sigma(G) \leq 2$ and $|\rho(G)| \geq 5$. Then $G/N$ is an almost simple group.
Proof. We first claim that, if $M/N$ is a chief factor of $G$, then $M/N$ is a nonabelian simple group.

Let $M$ be a normal subgroup of $G$ such that $M/N$ is a chief factor of $G$. Since $N$ is the largest normal solvable subgroup of $G$, we deduce that $M/N$ is nonsolvable so that $M/N \cong S^k$ for some integer $k \geq 1$ and some nonabelian simple group $S$. Let $C/N = C_{G/N}(M/N)$. Then $G/C$ embeds into $\text{Aut}(S^k)$.

Assume first that $k \geq 3$. Since $|\rho(S)| = |\pi(S)| \geq 3$, there exist three distinct prime divisors $r_i$, $1 \leq i \leq 3$, and three characters $\psi_i \in \text{Irr}(S)$ for $1 \leq i \leq 3$ with $r_i | \psi_i(1)$. Let

$$\varphi = \psi_1 \times \psi_2 \times \psi_3 \times 1 \times \cdots \times 1 \in \text{Irr}(S^k).$$

Then $\sigma(\varphi) \geq 3$, which is a contradiction since

$$\sigma(S^k) = \sigma(M/N) \leq \sigma(M) \leq \sigma(G) \leq 2.$$ 

Thus $k \leq 2$.

Now assume that $k = 2$. Let $B/C = (G/C) \cap \text{Aut}(S)^2$. Then $G/B$ is a nontrivial subgroup of the symmetric group of degree 2, and thus $|G:B| = 2$. Since $S^2 \cong MC/C \leq B/C \leq G/C$ and $\sigma(G) \leq 2$, we deduce that $\sigma(S^2) \leq 2$, and thus $\sigma(S) = 1$, by Lemma 3.1. By [12, Satz 8], we know that $S$ is isomorphic to either $\text{PSL}_2(4)$ or $\text{PSL}_2(8)$. In both cases, we obtain that $\pi(\text{Aut}(S)) = \pi(S)$; hence, $\pi(B/C) = \pi(S)$. Moreover, as $|G:B| = 2$, we deduce that $\pi(G/C) = \pi(S)$. As $G/C$ has no nontrivial normal abelian Sylow subgroups, Ito-Michler’s theorem yields that $\rho(G/C) = \pi(G/C) = \pi(S)$. Since $|\pi(G/C)| = |\pi(S)| = 3$ and $|\rho(G)| \geq 5$, there exists $r \in \rho(G) - \pi(G/C)$. Then $r > 2$ and $r \in \rho(C)$ by Lemma 3.2. Let $\theta \in \text{Irr}(C)$ be such that $r | \theta(1)$. Let $L$ be a normal subgroup of $MC$ such that $L/C \cong S$. Notice that $MC/C \cong S^2$. By applying [19, Lemma 4.2], $\theta$ extends to $\theta_0 \in \text{Irr}(L)$. By Gallagher’s theorem [9, Corollary 6.17], $\theta_0 \mu \in \text{Irr}(L)$ for all $\mu \in \text{Irr}(L/C)$. Let $\mu_0 \in \text{Irr}(L/C)$ with $2 | \mu_0(1)$, and let $\varphi = \theta_0 \mu_0 \in \text{Irr}(L)$. Then $\pi(\varphi(1)) = \{2, r\}$ with $r > 2$. As $MC/L \cong S$, we can apply [19, Lemma 4.2] again to obtain that $\varphi$ extends to $\varphi_0 \in \text{Irr}(MC)$ and then, by applying Gallagher’s theorem, $\varphi_0 \mu \in \text{Irr}(MC)$ for all $\mu \in \text{Irr}(MC/L)$. Clearly, $MC/L \cong S$ has an irreducible character $\tau \in \text{Irr}(MC/L)$ with $s | \tau(1)$, where $s \notin \{2, r\}$. We now have that $\varphi_0 \tau \in \text{Irr}(MC)$. But then this is a contradiction.
as \( \pi(\varphi_0(1)\tau(1)) = \{2, s, r\} \). This contradiction shows that \( k = 1 \), as wanted.

Let \( M/N \) be a chief factor of \( G \), and let \( C/N = C_{G/N}(M/N) \). We claim that \( C = N \) and thus \( G/N \) is an almost simple group as required. By the claim above, we know that \( M/N \cong S \) for some nonabelian simple group \( S \). Hence, \( G/C \) is an almost simple group with socle \( MC/C \cong M/N \). Suppose, by contradiction, that \( C \neq N \). Now let \( L/N \) be a chief factor of \( G \) with \( N \leq L \leq C \). By the claim above, we deduce that \( L/N \) is isomorphic to some nonabelian simple group. In particular, \( |\rho(C/N)| \geq |\pi(L/N)| \geq 3 \). We have that \( MC/N \cong C/N \times M/N \). Since \( \sigma(MC/N) \leq \sigma(MC) \leq \sigma(G) \leq 2 \), we deduce that \( \sigma(C/N \times M/N) \leq 2 \) and thus by Lemma 3.1, \( \sigma(C/N) = 1 = \sigma(M/N) \). By [12], we have \( C/N \cong T \times A \), where \( A \) is abelian, \( T \) is a nonabelian simple group and \( S, T \in \{ \text{PSL}_2(4), \text{PSL}_2(8) \} \). Since \( C \leq G \) and the solvable radical \( W \) of \( C \) is characteristic in \( C \), we obtain that \( W \leq G \), and thus \( W \leq N \) as \( N \) is the largest normal solvable subgroup of \( G \). Clearly, \( N \leq W \) as \( N \) is also a solvable normal subgroup of \( C \), so \( W = N \). Therefore, \( C/N \) has no nontrivial normal abelian subgroup. Thus, \( A = 1 \), and hence \( C/N \cong T \). Since \( \pi(G/C) = \pi(M/N) \) and \( G/N \) has no normal abelian Sylow subgroup, we obtain that

\[
\rho(G/N) = \pi(G/N) = \pi(C/N) \cup \pi(M/N) = \pi(S) \cup \pi(T).
\]

It follows that

\[
|\rho(G/N)| = |\pi(S) \cup \pi(T)| \leq |\pi(\text{PSL}_2(4)) \cup \pi(\text{PSL}_2(8))| = 4.
\]

Hence, \( \rho(G) - \rho(G/N) \) is nonempty. Now let \( r \in \rho(G) - \rho(G/N) \). As \( \{2, 3\} \subseteq \rho(G/N) \), we obtain that \( r \notin \{2, 3\} \). By Lemma 3.2, \( r \in \rho(N) \), and hence \( r \mid \theta(1) \) for some \( \theta \in \text{Irr}(N) \). Since \( \sigma(M) \leq \sigma(G) \leq 2 \) and \( M/N \cong S \), by [19, Lemma 4.2], we deduce that \( \theta \) extends to \( \theta_0 \in \text{Irr}(M) \). Now let \( \lambda \in \text{Irr}(M/N) \) with \( 2 \mid \lambda(1) \). By Gallagher’s theorem, \( \varphi = \theta_0\lambda \in \text{Irr}(M) \) with \( \pi(\varphi(1)) = \{2, r\} \). Notice that \( r \geq 5 \) since \( r \notin \{2, 3\} \). Now let \( K = MC \leq G \). Then \( K/M \cong T \) and \( \sigma(K) \leq 2 \). Applying the same argument as above, we deduce that \( \varphi \) extends to \( \varphi_0 \in \text{Irr}(K) \). Clearly, \( K/M \cong T \) has an irreducible character \( \mu \) with \( 3 \mid \mu(1) \) and thus, by Gallagher’s theorem again, \( \psi = \varphi_0\mu \in \text{Irr}(K) \) and obviously \( \sigma(\psi) \geq 3 \), which is a contradiction. \( \square \)

We are now ready to prove Theorem B, which we state here.
Theorem 3.4. Let $G$ be a group. If $\sigma(G) \leq 2$, then $|\rho(G)| \leq 2\sigma(G)+1$.

Proof. Let $G$ be a counterexample to the theorem with minimal order. Then $\sigma(G) \leq 2$, but $|\rho(G)| > 2\sigma(G) + 1$. If $G$ is solvable or $G$ is nonsolvable with $\sigma(G) = 1$, then

$$|\rho(G)| \leq 2\sigma(G) + 1$$

by [6, 11, 12], which is a contradiction. Thus, we can assume that $G$ is nonsolvable, $\sigma(G) = 2$ and $|\rho(G)| \geq 6$. Let $N$ be the solvable radical of $G$. By Lemma 3.3, $G/N$ is an almost simple group with simple socle $M/N$. Since $\sigma(M/N) \leq \sigma(G/N) \leq \sigma(G) = 2$, we deduce from Lemmas 2.5 and 2.4 that

$$|\pi(G/N)| = |\pi(M/N)| \leq 5.$$ 

As $|\rho(G)| \geq 6$, we have that $\rho(G) - \rho(G/N)$ is nonempty and let $r \in \rho(G) - \rho(G/N)$. By Lemma 3.2, $r | \theta(1)$ for some $\theta \in \text{Irr}(N)$. Since $\sigma(M) \leq 2$, by applying [19, Lemma 4.2], we deduce that $\theta$ extends to $\theta_0 \in \text{Irr}(M)$. Using Gallagher’s theorem, we must have that $\sigma(M/N) = 1$, and hence $M/N \cong \text{PSL}_2(4)$ or $\text{PSL}_2(8)$. Thus, $|\pi(G/N)| = |\pi(M/N)| = 3$; hence, $|\tau| \geq 3$ with $\tau = \rho(G) - \rho(G/N)$. By Lemma 3.2, we have that $\tau \subseteq \rho(N)$ and, since $N$ is solvable, by applying Pálfy’s condition [16, Theorem], there exists $\psi \in \text{Irr}(N)$ such that $\psi(1)$ is divisible by two distinct primes in $\tau$. Now, by applying [19, Lemma 4.2] again, we obtain a contradiction. This contradiction shows that $|\rho(G)| \leq 2\sigma(G) + 1$, as wanted.

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