ON ALGEBRAS OF BANACH ALGEBRA-VALUED BOUNDED CONTINUOUS FUNCTIONS

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ABSTRACT. Let $X$ be a completely regular Hausdorff space. We denote by $C(X, A)$ the algebra of all continuous functions on $X$ with values in a complex commutative unital Banach algebra $A$. Let $C_b(X, A)$ be its subalgebra consisting of all bounded continuous functions and endowed with the uniform norm. In this paper, we give conditions equivalent to the density of a natural continuous image of $X \times \mathcal{M}(A)$ in the maximal ideal space of $C_b(X, A)$.

1. Introduction. Throughout this paper, $X$ will denote a completely regular Hausdorff space, $A$ a complex commutative unital Banach algebra with norm $\| \cdot \|$ and unit element $e$ and $G(A)$ the set of invertible elements of $A$. We may assume that $\|e\| = 1$. We shall use the following notation for various function spaces:

$C(X, A)$ is the unital algebra of all continuous functions on $X$ with values in $A$, with pointwise operations and unit element the function on $X$ identically equal to $e$ and which will be denoted simply by $e$.

$C_b(X, A)$ is the subalgebra of $C(X, A)$ of all bounded continuous functions, provided with the uniform norm $\| \cdot \|_\infty$ given by $\|f\|_\infty = \sup_{x \in X} \|f(x)\|$.

When $A$ is the complex field $\mathbb{C}$, then we shall write $C(X)$ and $C_b(X)$ instead of $C(X, \mathbb{C})$ and $C_b(X, \mathbb{C})$, respectively.

$C_p(X, A)$ is the subalgebra of $C_b(X, A)$ of all continuous functions $f$ such that the closure of its range in $A$, namely $\text{cl}(f(X))$, is compact in $A$.


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It is easy to see that $C_b(X, A)$ and $C_p(X, A)$ are Banach algebras. In general, $C_p(X, A)$ is a proper subalgebra of $C_b(X, A)$ as the next example shows. Take $X = \mathbb{N}$ endowed with the discrete topology and $A = C([0, 1])$ with the uniform norm. Let $f : X \to A$ be the function given by $f(n)(0) = f(n)(1) = 1$, $f(n)(1-1/n) = 1/n$ and $f(n)$ is linear elsewhere in $[0, 1]$. Then $f \in C_b(X, A) \setminus C_p(X, A)$, since the sequence $(f(n))$ has no uniformly convergent subsequence in $C([0, 1])$.

Necessary and sufficient conditions for the equality of the latter algebras are given in the next easily proven result.

**Proposition 1.1.** The following assertions are equivalent:

(i) $C_b(X, A) = C_p(X, A)$.
(ii) If $f \in C_b(X, A)$ and $f(X) \subset G(A)$, then $\text{cl}(f(X))$ is compact.
(iii) For every $f \in C_b(X, A)$, there exist $\lambda_1, \lambda_2 \in \mathbb{C}$, with $\lambda_1 \neq 0$, such that $\lambda_1 f + \lambda_2 e \in C_p(X, A)$.

For every $f \in C_p(X, A)$, there exists a unique extension $f_\beta$ of $f$ to the Stone-Čech compactification $\beta X$ of $X$.

Let $B$ be any complex commutative unital algebra. We denote by $\mathcal{M}^\#(B)$ the set of all non-zero multiplicative linear functionals on $B$, provided with the weak star topology $w^*$. When $B$ is a topological algebra, $\mathcal{M}(B)$ denotes the topological subspace of $\mathcal{M}^\#(B)$ consisting of all non-zero multiplicative continuous linear functionals on $B$. For $b \in B$, its Gelfand transform $\hat{b}$ is given by $\hat{b}(\varphi) = \varphi(b)$, for $\varphi \in \mathcal{M}^\#(B)$. The set $\mathcal{M}(B)$ is called the maximal ideal space of $B$ and it coincides with $\mathcal{M}^\#(B)$ if $B$ is a Banach algebra.

There are several papers in which $\mathcal{M}^\#(B)$ or $\mathcal{M}(B)$ is characterized when $B$ is a function algebra. Well-known results are: $\mathcal{M}^\#(C(X)) = X$ if $X$ is a realcompact space ([5, page 3609, Theorem 1]) and $\mathcal{M}(C_b(X)) = \beta(X)$ if $X$ is a completely regular Hausdorff space ([11, page 123, Theorem (3.2.11)])

Along these lines, Dierolf, Schröder and Wengenroth proved in [3, page 54, Theorem 1], the formula $\mathcal{M}^\#(C(X, E)) = X \times \mathcal{M}^\#(E)$ for a (completely regular Hausdorff) realcompact space $X$ and a metrizable topological algebra $E$. Under the same assumption on $X$ this formula was previously proved in [8, page 371, Theorem 5 (a)] by Hery
supposing that $E$ is a unital commutative topological $Q$-algebra with continuous inversion and either $\mathcal{M}(E)$ is locally equicontinuous or $X$ is discrete.

Concerning the maximal ideal spaces of functions algebras, Hausner in [7, page 248, Theorem], Dietrich in [4, page 207, Theorem 4] and Kahn in ([9, page 89, Theorem 5.2.4]) proved that $\mathcal{M}(C(X, E)) = X \times \mathcal{M}(E)$. In the first of these works $X$ is a compact Hausdorff space and $E$ is a unital complex commutative Banach algebra. In the second one, $X$ is any completely regular $k$-space and $E$ is a unital complete locally convex algebra such that $\mathcal{M}(E)$ is locally equicontinuous. In Kahn’s, $X$ is a completely regular space of finite covering dimension and $E$ is a unital topological algebra with non-trivial dual and such that $\mathcal{M}(E)$ is locally equicontinuous. In all these papers $C(X, E)$ carries the compact-open topology. Using any of these results or [1, page 314, Corollary 6], the equality $\mathcal{M}(C_p(X, A)) = \beta X \times \mathcal{M}(A)$, which is a particular case of [8, page 369, Corollary 2 (a)], is easily obtained in Proposition 2.1 under our general hypothesis on $X$ and $A$.

In contrast, little is known in general about the maximal ideal space of $C_b(X, A)$. Govaerts showed in [6, page 156, Counterexample 1] that $\mathcal{M}(C_b(X, A)) = \beta X \times \mathcal{M}(A)$ is false in general, and Kahn proved in [9, page 89, Corollary 5.2.3] that $\mathcal{M}(C_b(X, E)) = X \times \mathcal{M}(E)$, where $C_b(X, E)$ is endowed with the strict topology for any completely regular space $X$ of finite covering dimension and a unital topological algebra $E$ with non-trivial dual for which $\mathcal{M}(E)$ is locally equicontinuous. The notion of the strict topology on $C_b(X, E)$ was first introduced by Buck in [2, page 97, Definition] in the case of $X$ locally compact and $E$ locally convex.

Here we study $\mathcal{M}(C_b(X, A))$. We define a natural continuous transformation $T$ from $X \times \mathcal{M}(A)$, with the product topology, into $\mathcal{M}(C_b(X, A))$. Therefore, each function $f \in C_b(X, A)$ has its proper Gelfand transform $\widehat{f} \in C(\mathcal{M}(C_b(X, A)))$ and also another Gelfand transform $\widetilde{f} = \widehat{f} \circ T$ belonging to $C_b(X \times \mathcal{M}(A))$. We prove that the transformation $f \to \widetilde{f}$ is a continuous homomorphism.

Let $A$ be a complex completely symmetric algebra, i.e., a complex commutative unital Banach algebra with involution $*$ satisfying $\|a\| = \|a^*\|$ and $F(a^*) = F(a)$ (the complex conjugate of $F(a)$) for all $a \in A$.
and $F \in \mathcal{M}(A)$. We show that property “$f \in C_b(X, A)$ is invertible if $\tilde{f}$ is invertible” is equivalent to “$T(X \times \mathcal{M}(A))$ is dense in $\mathcal{M}(C_b(X, A))$.”

We do not know if these two properties are still equivalent if $A$ is not assumed as above, but we exhibit an example, orally proposed by V. Müllner, in which $A$ is a complex completely symmetric algebra and nevertheless there exists $f \in C_b(X, A)$ such that $\tilde{f}$ is invertible and $f$ is not. Therefore even for completely symmetric algebras, the set $\mathcal{M}(C_b(X, A))$ is in general larger than the $w^*$-closure $\text{cl}_{w^*}(T(X \times \mathcal{M}(A)))$ of $T(X \times \mathcal{M}(A))$.

Any $C^*$-algebra is an example of a completely symmetric algebra ([10, page 233, Corollary 4]), but here we are not going to assume that the involution on $A$ satisfies $\|aa^*\| = \|a\|^2$, not even the weaker condition $\|aa^*\| = \|a\|\|a^*\|$, for $a \in A$.

2. Results. In this section, we define a natural continuous transformation $T$ from $X \times \mathcal{M}(A)$, with the product topology, into $\mathcal{M}(C_b(X, A))$ and through it and the classical Gelfand transform $\hat{f}$ for $f \in C_b(X, A)$, we introduce the Gelfand transform $\tilde{f}$ with respect to $X \times \mathcal{M}(A)$. Using $T$ and $\tilde{f}$, we shall state and prove almost all the results. In order to avoid confusion on the scope of these, we recall that we are assuming that $X$ is a completely regular Hausdorff space and $A$ is a complex commutative unital Banach algebra. From Lemma 2.5 on, $A$ is a complex completely symmetric algebra with continuous involution.

Proposition 2.1. The function $H : C_p(X, A) \to C(\beta X, A)$, with $H(f) = f_\beta$ is an isometric isomorphism of algebra and $\mathcal{M}(C_p(X, A)) = \beta X \times \mathcal{M}(A)$.

Proof. It is readily seen that $H$ is a bijective homomorphism of algebras. We also have that $\|f\|_\infty = \|f_\beta\|_\infty$, since $X$ is dense in $\beta X$ and then $H$ is an isometry. Thus, $\mathcal{M}(C_p(X, A)) = \mathcal{M}(C(\beta X, A))$. Since $\mathcal{M}(C(\beta X, A)) = \beta X \times \mathcal{M}(A)$, the result follows.

Proposition 2.2. There exists a continuous mapping $T$ from $X \times \mathcal{M}(A)$ into $\mathcal{M}(C_b(X, A))$, given by $T(x, F) = T_{(x, F)}$, where

$$T_{(x, F)}(f) = F(f(x)) = \tilde{f}(x)(F),$$
for every \( f \in C_b(X, A) \) and \( \hat{f}(x) \) is the Gelfand transform of \( f(x) \). This mapping \( T \) has a continuous extension \( T_\beta \) to \( \beta(X \times \mathcal{M}(A)) \).

**Proof.** It is clear that \( T(x,F) \in \mathcal{M}(C_b(X, A)) \). Given the \( w^* \)-neighborhood \( U = V(T(x,F), f_1, \ldots, f_n, \epsilon) \) of \( T(x,F) \) take the \( w^* \)-neighborhood \( W = V(F, f_1(x), \ldots, f_n(x), \epsilon/2) \) of \( F \) and a neighborhood \( V(x) \) of \( x \) satisfying \( \|f_i(x) - f_i(y)\| < \epsilon/2 \) if \( y \in V(x) \) and \( 1 \leq i \leq n \). Then, for \( (y,G) \in V(x) \times W \), we have that \( T(y,G) \in U \).

Since \( \mathcal{M}(C_b(X, A)) \) is compact, then \( T \) has a continuous extension \( T_\beta \) to \( \beta(X \times \mathcal{M}(A)) \).

**Corollary 2.3.** \( T_\beta(\beta(X \times \mathcal{M}(A))) = \text{cl}_{w^*}(T(X \times \mathcal{M}(A))) \).

**Proof.** Since \( T_\beta \) is continuous and \( X \times \mathcal{M}(A) \) is dense in \( \beta(X \times \mathcal{M}(A)) \), we get that \( T_\beta(\beta(X \times \mathcal{M}(A))) \subset \text{cl}_{w^*}(T(X \times \mathcal{M}(A))) \). But \( T_\beta(\beta(X \times \mathcal{M}(A))) \), being weak\(^*\)-compact, contains the weak\(^*\)-closure of \( T(X \times \mathcal{M}(A)) \).

Taking \( f \in C_b(X, A) \), we define its Gelfand’s transform \( \tilde{f} \) with respect to \( X \times \mathcal{M}(A) \) as \( \tilde{f} = \hat{f} \circ T \), i.e.,

\[
\tilde{f}(x,F) = F(f(x)),
\]

for \( (x,F) \in X \times \mathcal{M}(A) \). Therefore, \( \tilde{f} \in C_b(X \times \mathcal{M}(A)) \) and \( \|\tilde{f}\|_\infty \leq \|f\|_\infty \).

The mapping \( f \rightarrow \tilde{f} \) is a continuous homomorphism from \( C_b(X, A) \) into \( C_b(X \times \mathcal{M}(A)) \). Thus, if \( f \) is invertible in \( C_b(X, A) \), then \( \tilde{f} \) is invertible in \( C_b(X \times \mathcal{M}(A)) \).

The function \( \tilde{f} \) is invertible in the algebra \( C_b(X \times \mathcal{M}(A)) \) if and only if \( \tilde{f} \) is bounded away from zero, i.e., \( |F(f(x))| > \epsilon \) for some \( \epsilon > 0 \) and all \( (x,F) \in X \times \mathcal{M}(A) \). In particular, \( f \) is invertible in \( C(X, A) \) if \( \tilde{f} \) is invertible.

**Theorem 2.4.** For the following four assertions we have that: (i) implies (ii); (ii) implies (iv); and (ii) and (iii) are equivalent to each other.
(i) If \( f_1, \ldots, f_n \in C_b(X, A) \) and \( \epsilon > 0 \) are such that, for every \((x, F) \in X \times \mathfrak{M}(A)\), there exist \( 1 \leq i \leq n \) for which \( |f_i(x, F)| > \epsilon \), then there exist \( g_1, \ldots, g_n \in C_b(X, A) \) satisfying \( f_1g_1 + \cdots + f_ng_n = e \).

(ii) If \( f \in C_b(X, A) \) and \( \tilde{f} \) is invertible, then \( f \) is invertible.

(iii) If \( f \in C_b(X, A) \) and there exists \( \epsilon > 0 \) such that \( \|f(x) - y\| > \epsilon \) for all \( x \in X \) and \( y \in A \setminus G(A) \), then \( f \) is invertible.

(iv) If \( f \in C_b(X, A) \) and

\[
\sup \left\{ |\tilde{f}(x, F)| : (x, F) \in X \times \mathfrak{M}(A) \right\} < 1,
\]

then \( e - f \) is invertible.

**Proof.** Obviously, (i) implies (ii) and (ii) implies (iv).

(ii) \( \Rightarrow \) (iii). Assume that there exists \( \epsilon > 0 \) such that \( \|f(x) - y\| > \epsilon \) for all \( x \in X \) and \( y \in A \setminus G(A) \). Put \( y = f(x) - F(f(x))e \) for \( x \in X \) and \( F \in \mathfrak{M}(A) \). We have that \( y \notin G(A) \) and \( |\tilde{f}(x, F)| = |F(f(x))| = \|f(x) - y\| > \epsilon \), then \( \tilde{f} \) is invertible and, by (ii), \( f \) is invertible.

(iii) \( \Rightarrow \) (ii). Take \( f \in C_b(X, A) \), and suppose \( \tilde{f} \) is invertible. There exists an \( \epsilon > 0 \) such that \( |\tilde{f}(x, F)| > \epsilon \) for all \((x, F) \in X \times \mathfrak{M}(A)\). Given \( x \in X \) and \( y \in A \setminus G(A) \), choose \( F \in \mathfrak{M}(A) \) such that \( F(y) = 0 \) and put \( y = f(x) - F(f(x))e \). Then, \( \|f(x) - y\| = |\tilde{f}(x, F)| > \epsilon \); hence by (iii), \( f \) is invertible. \( \square \)

In the rest of this section we shall assume that \( A \) is a complex completely symmetric algebra with continuous involution \( * \).

**Lemma 2.5.** For every \( f \in C_b(X, A) \), there exists a \( g \in C_b(X, A) \) such that \( \tilde{g}(x, F) \) is the complex conjugate \( \overline{f(x, F)} \) of \( f(x, F) \) for each \((x, F) \in X \times \mathfrak{M}(A)\). Furthermore, we have \( |f|^2 = fg \).

**Proof.** If \( f \in C_b(X, A) \), then the function \( g \) defined by \( g(x) = f(x)^* \) belongs to \( C_b(X, A) \) because the involution is a continuous function. Then, we have

\[
\tilde{g}(x, F) = F(f(x)^*) = \overline{f(x, F)}
\]
and

\[ \tilde{f}g(x, F) = F(f(x) f(x)^*) = \left| \tilde{f}(x, F) \right|^2, \]

for all \((x, F) \in X \times \mathcal{M}(A)\).

\[ \square \]

**Theorem 2.6.** Assertions (i)–(iv) in Theorem 2.4 are all equivalent.

**Proof.**

(iv) \(\Rightarrow\) (ii). Take \(f \in C_b(X, A)\), and suppose that \(\tilde{f}\) is invertible. Then, \(\tilde{f}\) is bounded away from zero. Take \(g\) as in Lemma 2.5, and set \(M = \sup |\tilde{f}(x, F)|^2\) and \(N = \sup |e - (1/M)fg(x, F)|\), where the suprema are taken over all points \((x, F)\) in \(X \times \mathcal{M}(A)\). Since \(N = \sup |1 - (1/M)|\tilde{f}(x, F)|^2| < 1\), we have by (iv) that \((1/M)fg\) is invertible and then (ii) holds.

(ii) \(\Rightarrow\) (i). Suppose \(f_1, \ldots, f_n \in C_b(X, A)\) and \(\epsilon > 0\) are as in (i). Let \(g_i \in C_b(X, A)\) be such that \(|f_i|^2 = f_ig_i\) for every \(i = 1, 2, \ldots, n\). For \((x, F) \in X \times \mathcal{M}(A)\) we have that \(\sum_{i=1}^n |\tilde{f}_i(x, F)|^2 = \sum_{i=1}^n f_ig_i(x, F) = \sum_{i=1}^n f_ig_i(x, F) > \epsilon\). Thus, \(\sum_{i=1}^n f_ig_i\) is invertible in \(C_b(X \times \mathcal{M}(A))\). By (ii), \(\sum_{i=1}^n f_ig_i\) is invertible; therefore, there exists \(h \in C_b(X, A)\) such that \(\sum_{i=1}^n f_ig_ih = \epsilon\), that is, (i) holds.

\[ \square \]

**Proposition 2.7.** If \(T(X \times \mathcal{M}(A))\) is not dense in \(\mathcal{M}(C_b(X, A))\), then there exists an \(f \in C_b(X, A)\) such that \(\tilde{f}\) is invertible and \(f\) is not.

**Proof.** Let us assume that \(T(X \times \mathcal{M}(A))\) is not dense in \(\mathcal{M}(C_b(X, A))\), and take \(G \in \mathcal{M}(C_b(X, A))\setminus \text{cl}_{w^*}(T(X \times \mathcal{M}(A)))\). Then, there exist \(f_1, \ldots, f_n \in C_b(X, A)\) and \(\epsilon > 0\) such that, for each \((x, F) \in X \times \mathcal{M}(A)\), there is a \(1 \leq i \leq n\) such that \(|G(f_i) - F(f_i(x))| > \epsilon\). Put \(g_i = f_i - G(f_i)e\), and take \(h_i \in C_b(X, A)\) such that \(h_i(x, F) = \overline{g_i(x, F)}\) for \(1 \leq i \leq n\) and \((x, F) \in X \times \mathcal{M}(A)\). Then, for each \((x, F) \in X \times \mathcal{M}(A)\), \(|\overline{g_i(x, F)}| > \epsilon\) for some \(1 \leq i \leq n\) and \(G(g_i) = 0\) for all \(1 \leq i \leq n\).

Take

\[ f = \sum_{i=1}^n g_ih_i. \]
Then $G(f) = 0$ and

$$
|\tilde{f}(x, F)| = \sum_{i=1}^{n} |\tilde{g}_i(x, F)|^2 > \epsilon
$$

for all $(X, F) \in X \times \mathcal{M}(A)$. Therefore, $f$ is not invertible and $\tilde{f}$ is invertible. \qed

**Theorem 2.8.** Assertions (i)–(iv) of Theorem 2.4 are all equivalent to the following:

(v) $T(X \times \mathcal{M}(A))$ is dense in $\mathcal{M}(C_b(X, A))$.

**Proof.** From Proposition 2.7, (ii) implies (v). On the other hand, let us assume that $T(X \times \mathcal{M}(A))$ is dense in $\mathcal{M}(C_b(X, A))$ and take $f \in C_b(X, A)$ such that $\tilde{f}$ is invertible. Then there exists an $\epsilon > 0$ such that $|\tilde{f}(x, F)| > \epsilon$ for every $(x, F) \in X \times \mathcal{M}(A)$; hence, $\tilde{f}(G) \neq 0$ for all $G \in \mathcal{M}(C_b(X, A))$. Therefore, $f$ is invertible. \qed

**Corollary 2.9.** If $X$ is a pseudocompact space, then $T(X \times \mathcal{M}(A))$ is dense in $\mathcal{M}(C_b(X, A))$.

**Proof.** Suppose $f \in C_b(X, A)$ and $\tilde{f}$ is invertible. Then, $f$ is invertible in $C(X, A)$. Since the function $x \to ||f(x)^{-1}||$ is continuous in $X$, then it is bounded. Therefore, $f$ is invertible in $C_b(X, A)$. \qed

3. The example. We thank Vladimir Müller who orally communicated the next example to us that enables us to show that there is a completely symmetric algebra $A$ for which $T(N \times \mathcal{M}(A))$ is not dense in $\mathcal{M}(C_b(N, A))$.

Let $S$ be the free commutative group with countably many generators $a_1, a_2, \ldots$. Define a function $p : S \to (0, \infty)$ by $p(a_j^k) = 1$ for $k \geq 0$, $p(a_j^k) = j$ for $k < 0$ and $p(a_1^{k_1}a_2^{k_2}\cdots a_n^{k_n}) = p(a_1^{k_1})p(a_2^{k_2})\cdots p(a_n^{k_n})$. Then, $p$ is a positive multiplicative function.

Let $A$ be the weighted group algebra over $S$, i.e., $A$ is the set of functions $x : S \to \mathbb{C}$ satisfying that

$$
||x|| = \sum_{s \in S} |x(s)| p(s) < \infty,
$$
provided with the usual linear structure and the convolution product
\[(xy)(s) = \sum_{t \in S} x(t) y(t^{-1}s).\]

For each \(s \in S\), let \(\chi_s\) be the characteristic function of the singleton \(\{s\}\). Then, \(x = \sum_{s \in S} \alpha_s \chi_s\), with \(\alpha_s = x(s)\), for \(x \in A\). Identifying \(\chi_s\) with \(s\) in this expansion, we have
\[x = \sum_{s \in S} \alpha_s s,\]
\[\|x\| = \sum_{s \in S} |\alpha_s| p(s),\]
\[xy = \sum_{s \in S} \sum_{t \in S} \alpha_t \beta_{t^{-1}s}s,\]
if
\[x = \sum_{s \in S} \alpha_s s \quad \text{and} \quad y = \sum_{s \in S} \beta_s s,\]
and
\[F(x) = \sum_{s \in S} \alpha_s F(s) \quad \text{for every } F \in \mathcal{M}(A).\]

The algebra \(A\) under the involution defined by
\[\left(\sum_{s \in S} \alpha_s s\right)^* = \sum_{s \in S} \overline{\alpha_t}s\]
becomes a completely symmetric algebra.

If \(B = \{a_1,a_2,\ldots\}\), then clearly \(B \subset G(A)\) and \(B\) is a bounded set, keeping in mind that \(\|a_n\| = 1\) for each \(n\). Since \(A\) is a unital commutative Banach algebra, we have that \(\sigma(x) = \{F(x) : F \in \mathcal{M}(A)\}\) for each \(x \in A\). From this and applying the spectral radius formula to \(a_n\) and \(a_n^{-1}\), we have \(|F(a_n)| = 1\) for each \(n \in \mathbb{N}\) and \(F \in \mathcal{M}(A)\). Therefore, we have that \(\mathcal{M}(A) = S_1^\mathbb{N}\), associating each \(F \in \mathcal{M}(A)\) with the unique sequence \((e^{i\theta_1}, e^{i\theta_2}, \ldots)\) in the complex unit sphere \(S_1\) such that \(F(a_j) = e^{i\theta_j}\) for each \(j = 1,2,\ldots\).

Let us consider the algebra \(C_b(\mathbb{N},A)\) and the function \(f \in C_b(\mathbb{N},A)\) defined by \(f(n) = a_n\) for all \(n \geq 1\). Since \(|\hat{f}(n,F)| = 1\) for every \((n,F) \in \mathbb{N} \times \mathcal{M}(A)\), the function \(f\) is invertible. Nevertheless, \(f\) is not
invertible because \((f(\mathbb{N}))^{-1} = B^{-1}\) is not bounded. Therefore, we have that \(T(\mathbb{N} \times \mathcal{M}(A))\) is not dense in \(\mathcal{M}(C_b(\mathbb{N}, A))\). We point out that it can be shown that \(\sigma(f) = \{z : |z| \leq 1\}\).

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