THE DISCRIMINANT OF ABELIAN NUMBER FIELDS

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ABSTRACT. For an abelian number field $K$, the discriminant can be obtained from the conductor $m$ of $K$, the degree of $K$ over $\mathbb{Q}$, and the degrees of extensions $K \cdot \mathbb{Q}(\zeta_{m/p^\alpha})/\mathbb{Q}(\zeta_{m/p^\alpha})$, where $p$ runs through the set of primes that divide $m$, and $p^\alpha$ is the greatest power that divides $m$. In this paper, we give a formula for computing the discriminant of any abelian number field.

1. Introduction. Let $K$ be an algebraic number field over $\mathbb{Q}$. We say that $K$ is abelian if the extension field $K/\mathbb{Q}$ is Galois and $\text{Gal}(K/\mathbb{Q})$ is an abelian group. From the Kronecker-Weber theorem, if $K$ is an abelian number field, then $K$ is contained in some cyclotomic field $\mathbb{Q}(\zeta_m)$ [4]. The smallest $m$ for which this holds is called the conductor of $K$. The discriminant of a number field $K$, denoted $\text{disc}(K)$, is one of the most important invariants of an algebraic number field.

In the early 2000s, Interlando, Dantas Lopes and da Nobrega Neto presented a formula for computing the discriminant of subfields of $K = \mathbb{Q}(\zeta_{p^r})$, where $p$ is an odd prime and $r$ is a positive integer, see [3]. This formula depends only on $p$ and $[K : \mathbb{Q}]$. Later, in [1], Dantas Lopes extends the result of [3] to the case where $p = 2$, obtaining a formula that splits into two expressions depending on whether $K$ is cyclotomic or not. Both expressions depend only on $[K : \mathbb{Q}]$.

In a third work, Interlando et al., obtained a formula for computing the discriminant of an abelian number field $K$; this formula is presented as a function of the conductor $m$ of $K$, the degree $[K : \mathbb{Q}]$ and the degrees of the fields $K \cap \mathbb{Q}(\zeta_{m/p^\alpha})$ over $\mathbb{Q}$, where $p$ runs through the set of primes that divide $m$, and $p^\alpha$ runs through all powers of $p$ that...
divide $m$, see [2, Theorem 1]. In fact, they showed in their paper that it is not necessary to know all the powers of $p$ that divide $m$ but only the greatest power of $p$ which divides $m$, see [2, Theorem 2].

Unfortunately, [2, Theorem 2] is incorrect, since this result only applies to the case where $m$ is odd or a power of 2. The main purpose of this work is to correct this theorem. We present a formula when $m = 2^\alpha n$ with $\alpha, n \geq 2$ and $n$ odd. Furthermore, we show that, in [2, Theorem 1], the conductor can be omitted and, therefore, we can conclude that this result is a generalization of [1, Theorem 3.1] and [3, Theorem 4.1].

The paper is organized as follows. In Section 2, we briefly review the main theorems of [2] and show that, under certain conditions, the only subfields of $\mathbb{Q}(\zeta_{mp^n})$ are cyclotomic (Proposition 2.3). In Section 3, we show that, in [2, Theorem 1], the conductor can be omitted (Theorem 3.1), and this result is used to show that Theorem 2 of [2] is wrong. Finally, in Section 4, we give a corrected version of [2, Theorem 2] (Theorem 4.1).

2. Preliminaries. In this section, we present the main theorems of [2], proofs are omitted.

**Theorem 2.1 ([2, Theorem 1]).** Let

$$m = \prod_{i=1}^{k} p_i^{\alpha_i}$$

and $K$ be an abelian number field of conductor $m$. Then,

$$|\text{disc}(K)| = \frac{m^{[K: \mathbb{Q}]}}{\prod_{i=1}^{k} \sum_{r=1}^{\alpha_i} [K \cap \mathbb{Q}(\zeta_{m/p_i^r}): \mathbb{Q}]}.$$  

**Theorem 2.2 ([2, Theorem 2]).** Let $K$ be an abelian number field of conductor

$$m = \prod_{i=1}^{k} p_i^{\alpha_i}.$$
Then, the discriminant of $K$ is equal to

$$ |\text{disc}(K)| = \begin{cases} 
\left( \frac{m}{\prod_{i=1}^{k} p_i^{\alpha_i - 1 - (p_i - 1)/u_i}} \right) & \text{if } m \neq 2^n \\
2^{(n-1)2^{n-1}} & \text{if } K = \mathbb{Q}(\zeta_{2^n}) \\
2^{n2^{n-1} - 1} & \text{if } m = 2^{n+1} \\
& \text{for all } n \in \mathbb{N}; \\
& \text{for some } n \in \mathbb{N}, \\
& \text{but } K \neq \mathbb{Q}(\zeta_{2^{n+1}}),
\end{cases} $$

where $u_i = \left[ K : \mathbb{Q}(\zeta_{m/p_i^{\alpha_i}}) : \mathbb{Q}(\zeta_{m/p_i^{\alpha_i}}) \right]/p_i^{\alpha_i - 1}.$

In the proof of Theorem 2.2, Interlando et al. claim that,

if $m, n, p \in \mathbb{N}$ with $p$ a prime such that $(m, p^n) = 1,$
then, the only subfields of $\mathbb{Q}(\zeta_{mp^n})$ containing $\mathbb{Q}(\zeta_{mp})$
are the cyclotomic ones.

However, this claim is false when $p = 2,$ see Figures 1 and 2. Next, we
give the correct statement and its proof.

**Proposition 2.3.** Let $m, n, p \in \mathbb{N}$ with $p$ a prime such that $(m, p^n) = 1.$

Then,

1. If $p$ is odd, the only subfields containing $\mathbb{Q}(\zeta_{mp})$ and contained in
$\mathbb{Q}(\zeta_{mp^n})$ are cyclotomic.
2. If $p = 2,$ the only subfields containing $\mathbb{Q}(\zeta_{2mp^n})$ and contained in
$\mathbb{Q}(\zeta_{2mp^n})$ are cyclotomic.

**Proof.** We know that $\mathbb{Q}(\zeta_{mp^n}) = \mathbb{Q}(\zeta_{m})(\zeta_{p^n}),$ so that $\text{Gal}(\mathbb{Q}(\zeta_{mp^n})/\mathbb{Q}(\zeta_m))$ is isomorphic to a subgroup of $(\mathbb{Z}/p^n\mathbb{Z})^\times.$ Moreover, $\mathbb{Q}(\zeta_{mp^n})/\mathbb{Q}(\zeta_m)$ is a Galois extension, and therefore,

$$ |\text{Gal}(\mathbb{Q}(\zeta_{mp^n})/\mathbb{Q}(\zeta_m))| = [\mathbb{Q}(\zeta_{mp^n}) : \mathbb{Q}(\zeta_m)] = \phi(p^n) = |(\mathbb{Z}/p^n\mathbb{Z})^\times|. $$

This implies that $\text{Gal}(\mathbb{Q}(\zeta_{mp^n})/\mathbb{Q}(\zeta_m)) \cong (\mathbb{Z}/p^n\mathbb{Z})^\times.$ Thus, we have two cases:
(1) If \( p \) is odd, \( \mathbb{Q}(\zeta_{mp^n})/\mathbb{Q}(\zeta_m) \) is a cyclic extension of degree \( \phi(p^n) \). It follows that, for each divisor \( d \) of \( \phi(p^n) \) there is only one subfield of \( \mathbb{Q}(\zeta_{mp^n}) \) of degree \( d \) over \( \mathbb{Q}(\zeta_m) \). Furthermore, if \( E_i \) is a subfield such that
\[
\mathbb{Q}(\zeta_{mp}) \subset E_i \subset \mathbb{Q}(\zeta_{mp^n}),
\]
then
\[
[E_i : \mathbb{Q}(\zeta_m)] = [E_i : \mathbb{Q}(\zeta_{mp})][\mathbb{Q}(\zeta_{mp}) : \mathbb{Q}(\zeta_m)] = (p - 1)p^i
\]
for some \( i \in \{0, 1, \ldots, n - 1\} \). On the other hand,
\[
[\mathbb{Q}(\zeta_{mp^{i+1}}) : \mathbb{Q}(\zeta_m)] = (p - 1)p^i
\]
for each \( i \in \{0, 1, \ldots, n - 1\} \). Then, the uniqueness for degrees implies that \( E_i = \mathbb{Q}(\zeta_{mp^{i+1}}) \).

(2) If \( p = 2 \), the result is trivial when \( n = 2 \). Suppose \( n \geq 3 \). We consider the automorphism \( \sigma_5 \in \text{Gal}(\mathbb{Q}(\zeta_{m2^n})/\mathbb{Q}(\zeta_m)) \), given for \( \sigma_5(\zeta_2^n) = \zeta_2^5 \). Then, we have
\[
\sigma_5(\zeta_2^{2n-2}) = \zeta_2^{5\cdot2^{n-2}} = \zeta_2^{(1 + 2^2)2^{n-2}} = \zeta_2^{2^{n-2}},
\]
that is, \( \sigma_5 \) fixed to \( \zeta_2^{2n-2} = \zeta_{2^2} \), so that
\[
\langle \sigma_5 \rangle \subset \text{Gal}(\mathbb{Q}(\zeta_{m2^n})/\mathbb{Q}(\zeta_{m2^n})).
\]
On the other side, the extension \( \mathbb{Q}(\zeta_{m2^n})/\mathbb{Q}(\zeta_{m2^2}) \) is Galois, so that
\[
|\langle \sigma_5 \rangle| = 2^{n-2} = [\mathbb{Q}(\zeta_{m2^n}) : \mathbb{Q}(\zeta_{m2^n})] = |\text{Gal}(\mathbb{Q}(\zeta_{m2^n})/\mathbb{Q}(\zeta_{m2^n}))|.
\]
Hence, \( \mathbb{Q}(\zeta_{m2^n})/\mathbb{Q}(\zeta_{m2^2}) \) is a cyclic extension of degree \( 2^{n-2} \). It follows that a unique subfield \( E_i \) of \( \mathbb{Q}(\zeta_{m2^n}) \) exists such that \( [E_i : \mathbb{Q}(\zeta_{m2^2})] = 2^i \) for each \( i \in \{0, 1, \ldots, n - 2\} \). But, for \( i \in \{0, 1, \ldots, n - 2\} \), \( \mathbb{Q}(\zeta_{m2^{i+1}}) \) is a subfield with such properties. Then \( E_i = \mathbb{Q}(\zeta_{m2^{i+1}}) \).

**Corollary 2.4.** Let \( K \) be an abelian number field of conductor \( m = 2^{n+1} \). Then \( [K : \mathbb{Q}] = 2^n \) or \( 2^{n-1} \).

**Proof.** If \( m = 2^2 \), then \( K = \mathbb{Q}(\zeta_{2^2}) \). Thus, we consider the case \( m = 2^{n+1} \) with \( n \geq 2 \) and assume that \( K \neq \mathbb{Q}(\zeta_{2^{n+1}}) \), i.e., \( [K : \mathbb{Q}] \neq 2^n \) so that \( K \neq \mathbb{Q}(\zeta_{2^i}) \) for each \( i \in \{2, 3, \ldots, n\} \) since the conductor of \( K \) is \( m = 2^{n+1} \). Proposition 2.3 guarantees that \( K \cap \mathbb{Q}(\zeta_{2^2}) = \mathbb{Q} \) since only the cyclotomics contain \( \mathbb{Q}(\zeta_{2^2}) \); also, \( K \cdot \mathbb{Q}(\zeta_{2^2}) \) is a cyclotomic subfield
of \( \mathbb{Q}(\zeta_{2n+1}) \) containing \( K \). It follows that \( K \cdot \mathbb{Q}(\zeta_{2}) = \mathbb{Q}(\zeta_{2n+1}) \). Thus, if \( [K : \mathbb{Q}] = 2^r \), then

\[
2^{n-r} = [\mathbb{Q}(\zeta_{2n+1}) : K] = [K \cdot \mathbb{Q}(\zeta_{2}) : K] = [\mathbb{Q}(\zeta_{2}) : K \cap \mathbb{Q}(\zeta_{2})] = [\mathbb{Q}(\zeta_{2}) : \mathbb{Q}] = 2,
\]

and therefore, \( r = n - 1 \).

3. Removing the condition over the conductor. We generalize Theorem 2.1 to note that we can eliminate the condition over the conductor.

**Theorem 3.1.** Let \( K \) be a subfield of \( \mathbb{Q}(\zeta_{n}) \) with \( n = \prod_{j=1}^{s} p_j^{\beta_j} \). Then,

\[
|\text{disc}(K)| = \frac{n^{[K: \mathbb{Q}]} \prod_{j=1}^{s} \sum_{i=1}^{\beta_j} [K \cap \mathbb{Q}(\zeta_{n/p_j}) : \mathbb{Q}]}{\prod_{j=1}^{s} p_j^{\beta_j}}.
\]

**Proof.** If \( K \) is a subfield of \( \mathbb{Q}(\zeta_{n}) \) with \( n = \prod_{j=1}^{s} p_j^{\beta_j} \) and conductor \( m = \prod_{i=1}^{k} p_i^{\alpha_i} \) with \( m \neq n \), then \( k < s \) or \( \alpha_i < \beta_i \) for some \( i \in \{1, 2, \ldots, k\} \). We claim that

(3.1) \( \sum_{l=1}^{\beta_j} [K \cap \mathbb{Q}(\zeta_{n/p_j}) : \mathbb{Q}] = (\beta_j - \alpha_j) [K : \mathbb{Q}] + \sum_{i=1}^{\alpha_j} [K \cap \mathbb{Q}(\zeta_{m/p_j}) : \mathbb{Q}] \)

for \( 1 \leq j \leq k \), and

(3.2) \( \sum_{l=1}^{\beta_j} [K \cap \mathbb{Q}(\zeta_{n/p_j}) : \mathbb{Q}] = \beta_j [K : \mathbb{Q}] \)

for \( k + 1 \leq j \leq s \). To see this, we consider the following two cases.

(i) For \( 1 \leq j \leq k \), we have:

(a) if \( 1 \leq l \leq \beta_j - \alpha_j \), then \( m \mid n/p_j \), and, therefore,

\( K \subset \mathbb{Q}(\zeta_{m}) \subset \mathbb{Q}(\zeta_{n/p_j}) \),
so that

\[ [K \cap Q(\zeta_{n/p_j^l}) : Q] = [K : Q]. \]

(b) If \( \beta_j - \alpha_j + 1 \leq l \leq \beta_j \), then we have \( m/p_j^{l-\beta_j+\alpha_j} \mid n/p_j^l \),

Thus, \( Q(\zeta_{m/p_j^{l-\beta_j+\alpha_j}}) \subset Q(\zeta_{n/p_j^l}) \), and, therefore,

\[ K \cap Q(\zeta_{m/p_j^{l-\beta_j+\alpha_j}}) \subset K \cap Q(\zeta_{n/p_j^l}). \]

Conversely, observe that

\[ K \cap Q(\zeta_{n/p_j^l}) \subset Q(\zeta_m) \cap Q(\zeta_{n/p_j^l}) = Q(\zeta_{m,n/p_j^l}) = Q(\zeta_{m/p_j^{l-\beta_j+\alpha_j}}). \]

Then, \( K \cap Q(\zeta_{n/p_j^l}) \subset K \cap Q(\zeta_{m/p_j^{l-\beta_j+\alpha_j}}) \), and, therefore,

\[ K \cap Q(\zeta_{n/p_j^l}) = K \cap Q(\zeta_{m/p_j^{l-\beta_j+\alpha_j}}). \]

In conclusion,

\[
\sum_{l=1}^{\beta_j} [K \cap Q(\zeta_{n/p_j^l}) : Q] = \sum_{l=1}^{\beta_j-\alpha_j} [K \cap Q(\zeta_{n/p_j^l}) : Q] \\
+ \sum_{l=\beta_j-\alpha_j+1}^{\beta_j} [K \cap Q(\zeta_{n/p_j^l}) : Q] \\
= \sum_{l=1}^{\beta_j-\alpha_j} [K : Q] \\
+ \sum_{l=\beta_j-\alpha_j+1}^{\beta_j} [K \cap Q(\zeta_{m/p_j^{l-\beta_j+\alpha_j}}) : Q] \\
= (\beta_j - \alpha_j)[K : Q] \\
+ \sum_{i=1}^{\alpha_j} [K \cap Q(\zeta_{m/p_j^i}) : Q].
\]

(ii) For \( k + 1 \leq j \leq s \), similarly to case (ia), we have

\[ [K \cap Q(\zeta_{n/p_j^l}) : Q] = [K : Q] \quad \text{for } 1 \leq l \leq \beta_j. \]
Thus,
\[
\sum_{l=1}^{\beta_j} [K \cap \mathbb{Q}(\zeta_{n/p_j})] : \mathbb{Q} = \sum_{l=1}^{\beta_j} (K : \mathbb{Q}) = \beta_j [K : \mathbb{Q}].
\]

From equations (3.1), (3.2) and Theorem 2.1, we obtain
\[
\prod_{j=1}^{s} \frac{\beta_j [K : \mathbb{Q}]}{\sum_{i=1}^{\beta_j} [K \cap \mathbb{Q}(\zeta_{n/p_j})] : \mathbb{Q}} = \prod_{j=1}^{k} \frac{\beta_j [K : \mathbb{Q}] - \sum_{j=1}^{\beta_j} [K \cap \mathbb{Q}(\zeta_{n/p_j})] : \mathbb{Q}}{\sum_{i=1}^{\beta_j} [K \cap \mathbb{Q}(\zeta_{n/p_j})] : \mathbb{Q}}
\]
\[
= \prod_{j=1}^{k} \frac{\beta_j [K : \mathbb{Q}] - (\beta_j - \alpha_j) [K : \mathbb{Q}] - \sum_{i=1}^{\alpha_j} [K \cap \mathbb{Q}(\zeta_{m/p_j})] : \mathbb{Q}}{\sum_{i=1}^{\beta_j} [K \cap \mathbb{Q}(\zeta_{n/p_j})] : \mathbb{Q}}
\]
\[
= |\text{disc} (K)|.
\]

**Remark 3.2.** From Theorem 3.1, we conclude that Theorem 2.1 is, indeed, a generalization of previous works [1, 3].

Next, we compute the discriminant of two particular abelian number fields; these examples show that Theorem 2.2 is incorrect.

**Example 3.3.** The cyclotomic field \(\mathbb{Q}(\zeta_{24})\) has degree 8 over \(\mathbb{Q}\) and a Galois group \(G = \text{Gal}(\mathbb{Q}(\zeta_{24})/\mathbb{Q}) \cong (\mathbb{Z}/24\mathbb{Z})^\times\). We have
\[
G = \{\sigma_1, \sigma_5, \sigma_7, \sigma_{11}, \sigma_{13}, \sigma_{17}, \sigma_{19}, \sigma_{23}\},
\]
where \(\sigma_i(\zeta_{24}) = \zeta_{24}^i\). We consider the following subgroups of \(G\):
\[
H_J = \{\sigma_1, \sigma_7, \sigma_{17}, \sigma_{23}\},
H_K = \{\sigma_1, \sigma_7\},
H_L = \{\sigma_1, \sigma_7, \sigma_{13}, \sigma_{19}\},
H_N = \{\sigma_1, \sigma_{17}\},
H_O = \{\sigma_1, \sigma_{13}\},
\]
where \(J, K, L, N\) and \(O\) are the subfields of \(\mathbb{Q}(\zeta_{24})\) fixed by \(H_J, H_K, H_L, H_N\) and \(H_O\), respectively. We observe that \(L = \mathbb{Q}(\zeta_3) = \mathbb{Q}(\zeta_6)\), since \(\sigma_i(\zeta_3) = \sigma_i(\zeta_{24}^8) = \zeta_{24}^8 = \zeta_3\) for all \(\sigma_i \in H_L\); in a similar way, it is easily verified that \(N = \mathbb{Q}(\zeta_8)\) and \(O = \mathbb{Q}(\zeta_{12})\). For the subfield
Figure 1. Partial lattice of subfields of $\mathbb{Q}(\zeta_{24})$.

$K = \mathbb{Q}(\zeta_{24} + \zeta_{24}^{7})$, we have that

$$[K : \mathbb{Q}] = [\mathbb{Q}(\zeta_{24})]/|H_K| = 4,$$

and also

$$\langle H_K, H_N \rangle = H_J \implies K \cap \mathbb{Q}(\zeta_8) = J,$$

$$\langle H_K, H_O \rangle = H_L \implies K \cap \mathbb{Q}(\zeta_{12}) = L = \mathbb{Q}(\zeta_3),$$

$$\langle H_K, H_L \rangle = H_L \implies K \cap \mathbb{Q}(\zeta_6) = K \cap \mathbb{Q}(\zeta_3) = L = \mathbb{Q}(\zeta_3).$$

Figure 1 illustrates the partial lattice of subfields of $\mathbb{Q}(\zeta_{24})$. Then, by Theorem 3.1, we have that $|\text{disc}(K)| = 2^6 \cdot 3^2$. On the other hand, from Figure 1, we have that

$$[K \cdot \mathbb{Q}(\zeta_8) : \mathbb{Q}(\zeta_8)] = 2$$

and

$$[K \cdot \mathbb{Q}(\zeta_3) : \mathbb{Q}(\zeta_3)] = 2;$$

so, Theorem 2.2 says that $|\text{disc}(K)| = 2^7 \cdot 3^2$, a different answer to that obtained via Theorem 3.1.
Example 3.4. The cyclotomic field $\mathbb{Q}(\zeta_{40})$ has degree 16 over $\mathbb{Q}$ and a Galois group $G = \text{Gal}(\mathbb{Q}(\zeta_{40})/\mathbb{Q}) \cong (\mathbb{Z}/40\mathbb{Z})^*$. We have

$$G = \{\sigma_1, \sigma_3, \sigma_7, \sigma_9, \sigma_{11}, \sigma_{13}, \sigma_{17}, \sigma_{19}, \sigma_{21}, \sigma_{23}, \sigma_{27}, \sigma_{29}, \sigma_{31}, \sigma_{33}, \sigma_{37}, \sigma_{39}\},$$

where $\sigma_i(\zeta_{40}) = \zeta_{40}^i$. We consider the following subgroups of $G$:

$$H_J = \{\sigma_1, \sigma_{21}\},$$
$$H_K = \{\sigma_1, \sigma_{31}\},$$
$$H_L = \{\sigma_1, \sigma_{11}, \sigma_{21}, \sigma_{31}\},$$
$$H_M = \{\sigma_1, \sigma_9, \sigma_{17}, \sigma_{33}\},$$
$$H_N = \{\sigma_1, \sigma_7, \sigma_9, \sigma_{17}, \sigma_{23}, \sigma_{31}, \sigma_{33}, \sigma_{39}\},$$

where $J, K, L, M$ and $N$ are the subfields of $\mathbb{Q}(\zeta_{40})$ fixed by $H_J, H_K, H_L, H_M$ and $H_N$, respectively. We observe that $L = \mathbb{Q}(\zeta_{10}) = \mathbb{Q}(\zeta_5)$, $J = \mathbb{Q}(\zeta_{20})$, and $M = \mathbb{Q}(\zeta_8)$. For the subfield $K = \mathbb{Q}(\zeta_{40} + \zeta_{31}^{40})$ we have that $[K : \mathbb{Q}] = 8$, and also

$$\langle H_K, H_J \rangle = H_L \implies K \cap \mathbb{Q}(\zeta_{20}) = L,$$
$$\langle H_K, H_M \rangle = H_N \implies K \cap \mathbb{Q}(\zeta_8) = N,$$
$$\langle H_K, H_L \rangle = H_L \implies K \cap \mathbb{Q}(\zeta_{10}) = K \cap \mathbb{Q}(\zeta_5) = L = \mathbb{Q}(\zeta_5).$$

Figure 2 illustrates the partial lattice of subfields of $\mathbb{Q}(\zeta_{40})$. Then, by Theorem 3.1, we have that $|\text{disc}(K)| = 2^{12} \cdot 5^6$. On the other hand, from Figure 2, we have

$$[K \cdot \mathbb{Q}(\zeta_8) : \mathbb{Q}(\zeta_8)] = 4$$

and

$$[K \cdot \mathbb{Q}(\zeta_5) : \mathbb{Q}(\zeta_5)] = 2;$$

so, Theorem 2.2 says that $|\text{disc}(K)| = 2^{14} \cdot 5^6$, a different answer to that obtained via Theorem 3.1.

Abelian number fields presented in Examples 3.3 and 3.4 have the particularity that their conductor has the form $2^\alpha n$ with $\alpha, n \geq 2$ and $n$ odd. These examples suggest that Theorem 2.2 is incorrect. Other examples showed that similar problems arise when calculating
the discriminant of an abelian number field whose conductor is in the above-mentioned form.

4. Main result. In this section, we give a corrected version of Theorem 2.2 that works in general, including abelian number fields which have a conductor of form $m = 2^\alpha n$ with $\alpha, n \geq 2$, where $n$ is odd.

Theorem 4.1. Let $K$ be an abelian number field of conductor $m = \prod_{i=1}^l p_i^{\alpha_i}$. Then,

$$|\text{disc}(K)| = \left( \prod_{i=1}^l p_i^{\alpha_i - \lambda_i} \right)^{[K:Q]},$$

where

$$\lambda_i = \frac{p_i^{\alpha_i - (p_i, 2)} - 1 + (p_i - 1)/u_i}{p_i^{\alpha_i - (p_i, 2)} (p_i - 1)}$$
and
\[ u_i = \frac{[K \cdot \mathbb{Q}((\zeta_{m/p_i^{\alpha_i}})) : \mathbb{Q}(\zeta_{m/p_i^{\alpha_i}})]}{p_i^{\alpha_i-1}}. \]

Proof. From Theorem 2.1, it is sufficient to calculate the degrees \([K \cap \mathbb{Q}(\zeta_{m/p_i^{\alpha_i}}) : \mathbb{Q}]\) for all \(i \in \{1, 2, \ldots, l\}\) and \(1 \leq j \leq \alpha_i\). We have that \(\text{Gal}(K/K \cap \mathbb{Q}(\zeta_{m/p_i^{\alpha_i}})) \cong \text{Gal}(K \cdot \mathbb{Q}(\zeta_{m/p_i^{\alpha_i}})/\mathbb{Q}(\zeta_{m/p_i^{\alpha_i}}))\), and thus, \([K : K \cap \mathbb{Q}(\zeta_{m/p_i^{\alpha_i}})] = [K \cdot \mathbb{Q}(\zeta_{m/p_i^{\alpha_i}}) : \mathbb{Q}(\zeta_{m/p_i^{\alpha_i}})].\) It follows that
\[
[K \cap \mathbb{Q}(\zeta_{m/p_i^{\alpha_i}}) : \mathbb{Q}] = \frac{[K : \mathbb{Q}]}{[K : K \cap \mathbb{Q}(\zeta_{m/p_i^{\alpha_i}})]} = \frac{[K : \mathbb{Q}]}{[K \cdot \mathbb{Q}(\zeta_{m/p_i^{\alpha_i}}) : \mathbb{Q}(\zeta_{m/p_i^{\alpha_i}})]}.
\]

We have two cases:

(1) If \(p_i\) is odd, Proposition 2.3 says that the only intermediate subfields in extension \(\mathbb{Q}(\zeta_m)/\mathbb{Q}(\zeta_{m/p_i^{\alpha_i}})\) are cyclotomic. For \(1 \leq j < \alpha_i\), \(K \cdot \mathbb{Q}(\zeta_{m/p_i^{\alpha_i}})\) is a cyclotomic subfield of \(\mathbb{Q}(\zeta_m)\) containing \(K\). It follows that \(K \cdot \mathbb{Q}(\zeta_{m/p_i^{\alpha_i}}) = \mathbb{Q}(\zeta_m)\), so that
\[
[K \cdot \mathbb{Q}(\zeta_{m/p_i^{\alpha_i}}) : \mathbb{Q}(\zeta_{m/p_i^{\alpha_i}})] = [\mathbb{Q}(\zeta_m) : \mathbb{Q}(\zeta_{m/p_i^{\alpha_i}})] = \frac{\phi(m)}{\phi(m/p_i^{\alpha_i})} = p_i^j,
\]
and consequently,
\[
[K \cap \mathbb{Q}(\zeta_{m/p_i^{\alpha_i}}) : \mathbb{Q}] = \frac{[K : \mathbb{Q}]}{p_i^j}.
\]
If \(j = \alpha_i\), then \(K \cdot \mathbb{Q}(\zeta_{m/p_i^{\alpha_i}})\) has conductor \(m\) and
\[
[K \cdot \mathbb{Q}(\zeta_{m/p_i^{\alpha_i}}) : \mathbb{Q}(\zeta_{m/p_i^{\alpha_i}})] = \frac{[\mathbb{Q}(\zeta_m) : \mathbb{Q}(\zeta_{m/p_i^{\alpha_i}})]}{[\mathbb{Q}(\zeta_m) : K \cdot \mathbb{Q}(\zeta_{m/p_i^{\alpha_i}})]} = \frac{(p_i - 1)p_i^{\alpha_i - 1}}{[\mathbb{Q}(\zeta_m) : K \cdot \mathbb{Q}(\zeta_{m/p_i^{\alpha_i}})]};
\]
also, the extension \(\mathbb{Q}(\zeta_m)/\mathbb{Q}(\zeta_{m/p_i^{\alpha_i}})\) is cyclic and, therefore, \(p_i \nmid [\mathbb{Q}(\zeta_m) : K \cdot \mathbb{Q}(\zeta_{m/p_i^{\alpha_i}})]\). Otherwise, we have \(K \subset K \cdot \mathbb{Q}(\zeta_{m/p_i^{\alpha_i}}) \subset \mathbb{Q}(\zeta_{m/p_i}).\) Then
\[
[K \cdot \mathbb{Q}(\zeta_{m/p_i^{\alpha_i}}) : \mathbb{Q}(\zeta_{m/p_i^{\alpha_i}})] = u_i p_i^{\alpha_i - 1},
\]
and it follows that
\[ [K \cap \mathbb{Q}(\zeta_{m/p_i^{\alpha_i}}) : \mathbb{Q}] = \frac{[K : \mathbb{Q}]}{u_ip_i^{\alpha_i-1}}. \]

Next,
\[
\sum_{j=1}^{\alpha_i} [K \cap \mathbb{Q}(\zeta_{m/p_i^j}) : \mathbb{Q}] = [K : \mathbb{Q}] \left( \frac{1}{u_ip_i^{\alpha_i-1}} + \sum_{j=1}^{\alpha_i-1} \frac{1}{p_i^j} \right)
\]
\[
= [K : \mathbb{Q}] \left( \frac{1}{u_ip_i^{\alpha_i-1}} + \frac{1 - (1/p_i^{\alpha_i})}{1 - (1/p_i)} - 1 \right)
\]
\[
= [K : \mathbb{Q}] \left( \frac{p_i^{\alpha_i-1} - 1 + (p_i - 1)/u_i}{p_i^{\alpha_i-1}(p_i - 1)} \right)
\]
\[
= [K : \mathbb{Q}] \left( \frac{p_i^{\alpha_i-(p_i,2)} - 1 + (p_i - 1)/u_i}{p_i^{\alpha_i-(p_i,2)}(p_i - 1)} \right).
\]

(2) If \( p_i = 2 \), then \( \alpha_i \geq 2 \) and Proposition 2.3 say that only intermediate subfields in extension \( \mathbb{Q}(\zeta_m)/\mathbb{Q}(\zeta_{m/2}) \) are cyclotomic. For \( 1 \leq j < \alpha_i - 2 \), \( K \cdot \mathbb{Q}(\zeta_{m/2^j}) \) is a cyclotomic subfield of \( \mathbb{Q}(\zeta_m) \) containing \( K \). It follows that \( K \cdot \mathbb{Q}(\zeta_{m/2^j}) = \mathbb{Q}(\zeta_m) \), so that
\[ [K \cdot \mathbb{Q}(\zeta_{m/2^j}) : \mathbb{Q}(\zeta_m) : \mathbb{Q}(\zeta_{m/2^j})] = \frac{\phi(m)}{\phi(m/2^j)} = 2^j, \]
and, consequently,
\[ [K \cap \mathbb{Q}(\zeta_{m/2^j}) : \mathbb{Q}] = \frac{[K : \mathbb{Q}]}{2^j}. \]

If \( \alpha_i - 1 \leq j \leq \alpha_i \), then \( \mathbb{Q}(\zeta_{m/2^{\alpha_i-1}}) = \mathbb{Q}(\zeta_{m/2^{\alpha_i}}) \) and, therefore,
\[ [K \cap \mathbb{Q}(\zeta_{m/2^{\alpha_i-1}}) : \mathbb{Q}] = [K \cap \mathbb{Q}(\zeta_{m/2^{\alpha_i}}) : \mathbb{Q}] = \frac{[K : \mathbb{Q}]}{[K \cdot \mathbb{Q}(\zeta_{m/2^{\alpha_i}}) : \mathbb{Q}(\zeta_{m/2^{\alpha_i}})]}. \]

Then,
\[
\sum_{j=1}^{\alpha_i} [K \cap \mathbb{Q}(\zeta_{m/2^j}) : \mathbb{Q}] = [K : \mathbb{Q}] \left( \frac{2}{[K \cdot \mathbb{Q}(\zeta_{m/2^{\alpha_i}}) : \mathbb{Q}(\zeta_{m/2^{\alpha_i}})]} + \sum_{j=1}^{\alpha_i-2} \frac{1}{2^j} \right)
\]
\[= [K : \mathbb{Q}]\left(\frac{2^{\alpha_i-2} - 1 + 2^{\alpha_i-1} / [K \cdot \mathbb{Q}(\zeta_{m/2^{\alpha_i}}) : \mathbb{Q}(\zeta_{m/2^{\alpha_i}})]}{2^{\alpha_i-2}}\right)\]

\[= [K : \mathbb{Q}]\left(\frac{p_i^{\alpha_i-(p_i;2)} - 1 + (p_i - 1)/u_i}{p_i^{\alpha_i-(p_i;2)}(p_i - 1)}\right).\]

The result follows from Theorem 3.1.

We review Examples 3.3 and 3.4 and recalculate the discriminant of these abelian number fields using Theorem 4.1.

**Example 4.2.** In Example 3.3, we consider the abelian number field \(K = \mathbb{Q}(\zeta_{24} + \zeta_{24}^{-7})\); from Figure 1 and Theorem 4.1, it follows that

\[|\text{disc} (K)| = \left(\prod_{i=1}^{l} p_i^{\alpha_i-\lambda_i}\right)^{[K:\mathbb{Q}]} = (2^{3-(3/2)})^4 (3^{1-(1/2)})^4 = 2^6 \cdot 3^2.\]

This result is consistent with that obtained via Theorem 3.1.

In a similar way,

**Example 4.3.** In Example 3.4, we consider the abelian number field \(K = \mathbb{Q}(\zeta_{40} + \zeta_{40}^{31})\); from Figure 2 and Theorem 4.1, it follows that

\[|\text{disc} (K)| = \left(\prod_{i=1}^{l} p_i^{\alpha_i-\lambda_i}\right)^{[K:\mathbb{Q}]} = (2^{3-(3/2)})^8 (5^{1-(1/4)})^8 = 2^{12} \cdot 5^6.\]

This result is consistent with that obtained via Theorem 3.1.

The last two expressions in the formula of Theorem 2.2 can be derived from Theorem 4.1, as shown next.

**Corollary 4.4.** Let \(K\) be an abelian number field of conductor \(m = 2^{n+1}\). Then,

\[|\text{disc} (K)| = \begin{cases} 2^{n^2} & \text{if } K = \mathbb{Q}(\zeta_{2^{n+1}}); \\ 2^{n^2-1} - 1 & \text{otherwise}. \end{cases}\]
Proof. Since $[K \cdot \mathbb{Q}(\zeta_{m/2^n+1}) : \mathbb{Q}(\zeta_{m/2^n+1})] = [K : \mathbb{Q}]$, from Theorem 4.1 we obtain

$$|\text{disc}(K)| = \left(2^{n+1} - [2^{n-1} - 1 + (2^n/[K:Q])/2^{n-1}]\right)^{[K:Q]}$$

$$= 2^{[K:Q]((n2^{n-1}+1)/2^{n-1}) - 2}.$$ 

Moreover, from Corollary 2.4, we have

$$[K : \mathbb{Q}] = \begin{cases} 
2^n & \text{if } K = \mathbb{Q}(\zeta_{2^n+1}); \\
2^{n-1} & \text{otherwise,}
\end{cases}$$

and the result follows. \qed

Acknowledgments. The second author gratefully acknowledges the support given by CONACyT (Consejo Nacional de Ciencia y Tecnología) for the realization of this article. Finally, we thank the referee for his corrections and remarks.

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