MULTIPLE SOLUTIONS FOR A KIRCHHOFF-TYPE PROBLEM INVOLVING NONLOCAL FRACTIONAL p-LAPLACIAN AND CONCAVE-CONVEX NONLINEARITIES

CHANG-MU CHU, JIAO-JIAO SUN AND ZHI-PENG CAI

ABSTRACT. This paper examines a class of Kirchhoff nonlocal operators involving concave-convex nonlinearities and sign-changing weight functions. With the aid of the Nehari manifold, the existence of multiple nontrivial nonnegative solutions is obtained.

1. Introduction. In this paper, we study the multiplicity of nontrivial nonnegative solutions to the Dirichlet boundary value problem by the nonlocal operator:

\[
\begin{cases}
-M \left( \int_{\mathbb{R}^N} |u(x) - u(y)|^p \, dx \, dy \right) \mathcal{L}_p^p u \\
= \lambda f(x)|u|^{q-2}u + g(x)|u|^{r-2}u & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

where \( \Omega \subset \mathbb{R}^N \) is an open bounded set with Lipschitz boundary \( \partial \Omega \), \( M(t) = a + bt^m \) and the parameters \( a, b, \lambda > 0, 0 < s < 1 < q < p < \infty, N > ps \),

\[0 < m < \frac{ps}{N - ps}, \quad (m + 1)p < r < p_* = \frac{Np}{N - ps}.
\]

The weight functions \( f(x), g(x) \in C(\bar{\Omega}) \) satisfy \( f^+ = \max\{f, 0\} \neq 0 \) and \( g^+ = \max\{g, 0\} \neq 0 \). The nonlocal operator \( \mathcal{L}_p^p \) is defined as:

\[
\mathcal{L}_p^p u(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} |u(x) - u(y)|^{p-2}(u(x) - u(y))K(x-y) \, dy,
\]


This research was supported by the National Natural Science Foundation of China, grant No. 11661021 and by the Innovation Group Major Program of Guizhou Province, grant No. KY[2016]029. The first author is the corresponding author.

Received by the editors on March 10, 2016.
$x \in \mathbb{R}^N$, and $K : \mathbb{R}^N \setminus \{0\} \to (0, +\infty)$ is a measurable function with the property:

$$
\begin{align*}
\gamma K &\in L^1(\mathbb{R}^N) & \text{where } \gamma(x) = \min\{|x|^p, 1\}; \\
\text{there exists a } k_0 > 0 \text{ such that } K(x) &\geq k_0 |x|^{-(N + ps)} & \text{for any } x \in \mathbb{R}^N \setminus \{0\}; \\
K(x) &= K(-x) & \text{for any } x \in \mathbb{R}^N \setminus \{0\}.
\end{align*}
$$

A typical example for $K$ is given by the singular kernel $K(x) = |x|^{-(N + ps)}$. In this case, problem (1.1) becomes

$$
\begin{align*}
M \left( \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N + ps}} \, dx \, dy \right) (-\Delta)^s_{p} u \\
= \lambda f(x) |u|^{q-2} u + g(x) |u|^{r-2} u & \quad \text{in } \Omega, \\
u = 0 & \quad \text{in } \mathbb{R}^N \setminus \Omega,
\end{align*}
$$

where $(-\Delta)^s_{p}$ is the fractional $p$-Laplace operator which, up to normalization factors, may be defined as

$$
(-\Delta)^s_{p} u(x) = -2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_{\varepsilon}(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N + ps}} \, dy.
$$

Setting $m = 0$ and $a + b = 1$, problems (1.1) and (1.3) reduce to

$$
\begin{align*}
-\mathcal{L}^p_K u &= \lambda f(x) |u|^{q-2} u + g(x) |u|^{r-2} u & \quad \text{in } \Omega, \\
u &= 0 & \quad \text{in } \mathbb{R}^N \setminus \Omega,
\end{align*}
$$

and

$$
\begin{align*}
(-\Delta)^s_{p} u &= \lambda f(x) |u|^{q-2} u + g(x) |u|^{r-2} u & \quad \text{in } \Omega, \\
u &= 0 & \quad \text{in } \mathbb{R}^N \setminus \Omega.
\end{align*}
$$

Recently, much attention has been given to the study of Kirchhoff-type problems and nonlocal elliptic operators. Kirchhoff-type equations arise in the description of nonlinear vibrations of an elastic string, see [16]. Solvability of the Kirchhoff-type problem with the Dirichlet boundary using variational methods is studied in [1, 3, 7, 17], and the references therein. Fractional and nonlocal operators arise in many different contexts, such as the thin obstacle problem, finance, optimization, quasi-geostrophic flow, etc. There is much literature on
fractional and nonlocal operators; the interested reader is referred to [2, 4, 8, 9, 13]–[15, 18, 20, 22]–[26, 28], and the references therein. However, there is little research concerning the existence of solutions for Kirchhoff-type problems in the fractional setting. A detailed discussion regarding the physical meaning underlying fractional Kirchhoff models and their applications was first provided in [12, Appendix A]. More precisely, Fiscella and Valdinoci proposed a stationary Kirchhoff variational model, which takes into account the nonlocal aspects of tension arising from nonlocal measurements of the fractional length of the string. Some results of Kirchhoff-type problems involving nonlocal operators may also be found in [19, 21, 27], and the references therein.

In these papers, multiple results with concave-convex nonlinearities and sign changing weight functions were obtained using the Nehari manifold and fibering map analysis. However, as far as is known, there has been no research on Kirchhoff-type problems and nonlocal operators with concave-convex nonlinearities and sign changing weight functions.

Here, we use the variational approach on the Nehari manifold to solve problem (1.1). Motivated by [7, 13], the aim of this paper is to investigate multiple solutions of problem (1.1) and extend the results of [8, 13].

Problem (1.1) has variational structure, and solutions may be constructed as critical points of an associated energy functional on some appropriate space. In the norm $\|u\|_{H^s(\mathbb{R}^N)}$, the interaction between $\Omega$ and $\mathbb{R}^N \setminus \Omega$ provides a positive contribution, which should be considered when encoding the boundary condition $u = 0$ in $\mathbb{R}^N \setminus \Omega$ in the weak formulation.

Now, we introduce the linear space

$$X = \left\{ u \mid u : \mathbb{R}^N \rightarrow \mathbb{R} \text{ is measurable, } u|_\Omega \in L^p(\Omega) \right. \left. \text{and } (u(x) - u(y)) \sqrt{K(x - y)} \in L^p(Q) \right\},$$

where $Q = \mathbb{R}^{2N} \setminus (C\Omega \times C\Omega)$ with $C\Omega = \mathbb{R}^N \setminus \Omega$. The space $X$ is endowed with the norm, defined as

$$\|u\|_X = \|u\|_{L^p(\Omega)} + \left( \int_Q |u(x) - u(y)|^p K(x - y) \, dx \, dy \right)^{1/p}.$$
Moreover, we shall work in the closed linear subspace

\[ X_0 = \{ u \in X : u = 0 \text{ almost everywhere in } \mathbb{R}^N \setminus \Omega \}, \]

with the norm

\[ \| u \|_{X_0} = \left( \int_Q |u(x) - u(y)|^p K(x - y) \, dx \, dy \right)^{1/p}. \]

Let

\[ K = \mathbb{R}^N \setminus \{0\} \rightarrow (0, +\infty) \]

satisfy assumption (1.2). We have that \( C_0^\infty(\Omega) \subset X_0 \), and \((X_0, \| \cdot \|_{X_0})\) is a reflexive Banach space, see [10, 27]. Moreover,

\[ X \subset W^{s,p}(\Omega) \]

and

\[ X_0 \subset W^{s,p}(\mathbb{R}^N), \]

where \( W^{s,p}(\Omega) \) is the usual fractional Sobolev space endowed with the norm

\[ \| u \|_{W^{s,p}(\Omega)} = \| u \|_{L^p(\Omega)} + \left( \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n + ps}} \, dx \, dy \right)^{1/p}. \]

In addition, the embedding

\[ X_0 \hookrightarrow L^{p^*_s}(\Omega) \]

is continuous, and there exists a positive constant \( C_0 = C_0(N, p, s) \) such that, for any \( v \in X_0 \), \( 1 < k < p^*_s \),

\[ \| v \|_{L^k(\Omega)} \leq C_0 \| v \|_{X_0}. \]

Now, the definition of weak solutions for problem (1.1) is given.
Definition 1.1. We say that \( u \in X_0 \) is a weak solution of problem (1.1) if \( u \) satisfies
\[
M(\|u\|_{X_0}^p) \int_Q |u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))K(x-y) \, dx \, dy = \lambda \int_{\Omega} f(x)|u|^{q-2}u \varphi \, dx + \int_{\Omega} g(x)|u|^{r-2}u \varphi \, dx,
\]
for any \( \varphi \in X_0 \).

The main results of this paper are as follows.

Theorem 1.2. Let
\[
K : \mathbb{R}^N \setminus \{0\} \rightarrow (0, +\infty)
\]
be a function satisfying (1.2). Suppose that \( M(t) = a + bt^m \), \( f(x), g(x) \in C(\overline{\Omega}) \) satisfy
\[
f^+ = \max\{f, 0\} \neq 0 \quad \text{and} \quad g^+ = \max\{g, 0\} \neq 0.
\]
If \( 0 < s < 1 < q < p < \infty, N > ps, a > 0, b > 0, 0 \leq m < ps/(N-ps), (m+1)p < r < p^*_s \), then there exists a \( \lambda^* > 0 \) such that problem (1.1) for any \( \lambda \in (0, \lambda^*) \) has at least two nontrivial nonnegative solutions.

Remark 1.3. The multiple solutions of problem (1.4) with \( p = 2, f(x) = g(x) \equiv 1 \), are obtained in [8], and the multiple solutions of problem (1.5) are obtained in [13]. Obviously, the results there are the special case of our Theorem 1.2.

This paper is organized as follows. In Section 2, we give some notation and preliminaries regarding the Nehari manifold. In Section 3, we give the proof of Theorem 1.2.

2. Notation and preliminaries. The energy functional
\[
J_\lambda : X_0 \rightarrow \mathbb{R}
\]
associated to problem (1.1) is defined as
\[
J_\lambda(u) = \frac{1}{p} \widehat{M}(\|u\|_{X_0}^p) - \frac{\lambda}{q} \int_{\Omega} f(x)|u|^q \, dx - \frac{1}{r} \int_{\Omega} g(x)|u|^r \, dx,
\]
where  
\[
\widetilde{M}(t) = \int_0^t M(\tau) d\tau = at + \frac{b}{m+1} t^{m+1}.
\]

It is well known that \( J_\lambda \) is of class \( C^1 \) in \( X_0 \), and the solutions of problem (1.1) are critical points of the energy functional \( J_\lambda \) in \( X_0 \). In fact,

\[
\langle J'_\lambda(u), \varphi \rangle = M(\|u\|^p_{X_0}) \int_Q |u(x) - u(y)|^{p-2}(u(x) - u(y)) \\
\times (\varphi(x) - \varphi(y))K(x - y) \, dx \, dy \\
- \lambda \int_\Omega f(x)|u|^{q-2}u\varphi \, dx - \int_\Omega g(x)|u|^{r-2}u\varphi \, dx.
\]

Since \( r > (m+1)p \), it is easy to see that \( J_\lambda \) is unbounded from below on \( X_0 \). In order to obtain the existence results, we introduce the Nehari manifold

\[
\mathcal{N}_\lambda = \{ u \in X_0 \setminus \{0\} : \langle J'_\lambda(u), u \rangle = 0 \},
\]

where \( \langle , \rangle \) denotes the duality between \( X_0 \) and its dual space. Thus, \( u \in \mathcal{N}_\lambda \) if and only if

\[
M(\|u\|^p_{X_0})\|u\|^p_{X_0} - \lambda \int_\Omega f(x)|u|^{q} \, dx - \int_\Omega g(x)|u|^{r} \, dx = 0.
\]

It is clear that all nonzero solutions of problem (1.1) must lie on \( \mathcal{N}_\lambda \), and \( \mathcal{N}_\lambda \) is a much smaller set than \( X_0 \). Thus, it is easier to study \( J_\lambda \) on \( \mathcal{N}_\lambda \). The Nehari manifold \( \mathcal{N}_\lambda \) is closely linked to the behavior of functions of the form

\[
\psi_u : t \mapsto J_\lambda(tu) \quad \text{for} \ t > 0.
\]

Such maps are called fiber maps, introduced by Drabek and Pohozaev in [11] and discussed by Brown and Zhang in [6]. For \( u \in X_0 \), we have

\[
\psi_u(t) = \frac{1}{p} \widetilde{M}(t^p\|u\|^p_{X_0}) - \frac{\lambda t^q}{q} \int_\Omega f(x)|u|^{q} \, dx - \frac{t^r}{r} \int_\Omega g(x)|u|^{r} \, dx,
\]

\[
\psi'_u(t) = t^{p-1}M(t^p\|u\|^p_{X_0})\|u\|^p_{X_0} - \lambda t^{q-1} \int_\Omega f(x)|u|^{q} \, dx - t^{r-1} \int_\Omega g(x)|u|^{r} \, dx,
\]

\[
\psi''_u(t) = (p-1)t^{p-2}M(t^p\|u\|^p_{X_0})\|u\|^p_{X_0} + pt^{2p-2}M'(t^p\|u\|^p_{X_0})\|u\|^{2p}_{X_0}

- \lambda(q-1)t^{q-2} \int_\Omega f(x)|u|^{q} \, dx - (r-1)t^{r-2} \int_\Omega g(x)|u|^{r} \, dx.
\]
Clearly,

\[ t\psi_u'(t) = M(t^p\|u\|_{X_0}^p)\|tu\|_{X_0}^p - \lambda \int_{\Omega} f(x)|tu|^qdx - \int_{\Omega} g(x)|tu|^r dx, \]

and thus, for \( u \in X_0 \) and \( t > 0 \), \( \psi_u'(t) = 0 \) if and only if \( tu \in N_\lambda \). In particular, \( \psi_u'(1) = 0 \) if and only if \( u \in N_\lambda \). Therefore, it is natural to split \( N_\lambda \) into three parts corresponding to local minima, local maxima and points of inflection. For this, we set

\[
N_\lambda^+ = \{ u \in N_\lambda : \psi_u''(1) > 0 \},
\]

\[
N_\lambda^0 = \{ u \in N_\lambda : \psi_u''(1) = 0 \},
\]

\[
N_\lambda^- = \{ u \in N_\lambda : \psi_u''(1) < 0 \}.
\]

Thus, for each \( u \in N_\lambda \), we have

\[
\psi_u''(1) = pM'(||u||_{X_0}^p)||u||_{X_0}^{2p} \tag{2.2}
\]

\[
+ (p - q)M(||u||_{X_0}^p)||u||_{X_0}^p - (r - q) \int_{\Omega} g(x)|u|^r dx,
\]

or

\[
\psi_u''(1) = pM'(||u||_{X_0}^p)||u||_{X_0}^{2p} \tag{2.3}
\]

\[
- (r - p)M(||u||_{X_0}^p)||u||_{X_0}^p + \lambda(r - q) \int_{\Omega} f(x)|u|^q dx.
\]

Define

\[
\phi_\lambda(u) = \langle J'_\lambda(u), u \rangle = M(||u||_{X_0}^p)||u||_{X_0}^p \tag{2.4}
\]

\[
- \lambda \int_{\Omega} f(x)|u|^q dx - \int_{\Omega} g(x)|u|^r dx.
\]

Then, for \( u \in N_\lambda \),

\[
\langle \phi_\lambda'(u), u \rangle = (p - 1)M(||u||_{X_0}^p)||u||_{X_0}^p + pM'(||u||_{X_0}^p)||u||_{X_0}^{2p} \tag{2.5}
\]

\[
- (q - 1)\lambda \int_{\Omega} f(x)|u|^q dx - (r - 1) \int_{\Omega} g(x)|u|^r dx
\]

\[
= \psi_u''(1).
\]

Similar to the argument of [6, Theorem 2.3], we conclude the following result.
Lemma 2.1. Suppose that \( u_0 \) is a local minimizer for \( J_\lambda \) on \( N_\lambda \) and \( u_0 \notin N_\lambda^0 \). Then, \( J'_\lambda(u_0) = 0 \) in \( X_0^* \).

Proof. Since \( u_0 \) is a local minimizer on \( N_\lambda \) to \( J_\lambda \), by the theory of Lagrange multipliers and (2.4), there exists a \( \sigma \in \mathbb{R} \) such that

\[
J'_\lambda(u_0) = \sigma \phi'_\lambda(u_0).
\]

Thus,

\[
\langle J'_\lambda(u_0), u_0 \rangle = \sigma \langle \phi'_\lambda(u_0), u_0 \rangle = \sigma \psi''_{u_0}(1).
\]

Since \( u_0 \notin N_\lambda^0 \), we have \( \psi''_{u_0}(1) \neq 0 \). Hence, \( \sigma = 0 \), that is, \( J'_\lambda(u_0) = 0 \).

\( \blacksquare \)

Lemma 2.2. \( J_\lambda \) is coercive and bounded from below on \( N_\lambda \).

Proof. For \( u \in N_\lambda \), by (1.6) and (2.1),

\[
J_\lambda(u) = \frac{1}{p} M(||u||_{X_0}^p) - \frac{\lambda}{q} \int_{\Omega} f(x)|u|^q dx - \frac{1}{r} \int_{\Omega} g(x)|u|^r dx
\]

\[
= \frac{1}{p} M(||u||_{X_0}^p) - \frac{1}{r} M(||u||_{X_0}^p) ||u||_{X_0}^p - \frac{(r-q)\lambda}{rq} \int_{\Omega} f(x)|u|^q dx
\]

\[
\geq \frac{(r-p)a}{pr} ||u||_{X_0}^p + \frac{(r-(m+1)p)b}{(m+1)pr} ||u||_{X_0}^{(m+1)p}
\]

\[
- \frac{(r-q)\lambda}{rq} \|f^+\|_{\infty} C_0^q ||u||_{X_0}^q.
\]

Since \( 0 < q < 1 < p, (m+1)p < r < p_s^* \), we obtain that \( J_\lambda \) is coercive and bounded from below on \( N_\lambda \).

\( \blacksquare \)

Lemma 2.3. There exists a \( \lambda_1 > 0 \) such that \( N_\lambda^0 = \emptyset \) for any \( \lambda \in (0, \lambda_1) \).

Proof. We argue by contradiction. Assume that \( N_\lambda^0 \neq \emptyset \) for any \( \lambda > 0 \). Then, for \( u \in N_\lambda^0 \), we have \( \langle J'_\lambda(u), u \rangle = 0 \) and \( \psi''_{u_0}(1) = 0 \). In addition, from (2.2) and (2.3),

\[
(p-q)M(||u||_{X_0}^p) ||u||_{X_0}^p + pM'(||u||_{X_0}^p) ||u||_{X_0}^{2p} = (r-q) \int_{\Omega} g(x)|u|^r dx,
\]
and
\[(r - p)M(\|u\|^{p}_{X_0})\|u\|^{p}_{X_0} - pM'(\|u\|^{p}_{X_0})\|u\|^{2p}_{X_0} = \lambda(r - q)\int_{\Omega} f(x)|u|^q dx.\]

From (1.6),
\[((m + 1)p - q)b\|u\|^{(m+1)p}_{X_0} \leq C^p_0 (r - q)\|g^+\|_{\infty}\|u\|^{r}_{X_0},\]
and
\[(r - (m + 1)p)b\|u\|^{(m+1)p}_{X_0} \leq C^q_0 (r - q)\|f^+\|_{\infty}\|u\|^{q}_{X_0}.\]

This yields that
\[\left(\frac{(m+1)p-q}{C^p_0(r-q)\|g^+\|_{\infty}}\right)^{1/(r-(m+1)p)} \leq \|u\|_{X_0} \leq \left(\frac{(r-q)\|f^+\|_{\infty}C^q_0\lambda}{(r-(m+1)p)b}\right)^{1/(m+1)p-q}.\]

This is impossible if \(\lambda\) is sufficiently small. Thus, we obtain that a \(\lambda_1 > 0\) exists such that \(N^0_\lambda = \emptyset\) for any \(\lambda \in (0, \lambda_1).\)

By Lemma 2.3, for \(\lambda \in (0, \lambda_1),\) we write \(N_\lambda = N^{+}_\lambda \cup N^{-}_\lambda\) and define
\[a^+ = \inf_{u \in N^+_{\lambda}} J_\lambda(u), \quad a^- = \inf_{u \in N^-_{\lambda}} J_\lambda(u).\]

Let
\[
\lambda_2 = \frac{a(r - p)}{(r - q)C^q_0\|f^+\|_{\infty} - a (p - q)} \left( \frac{a (p - q)}{(r - q)C^p_0\|g^+\|_{\infty}} \right)^{(p-q)/(r-p)}.
\]

Then, we have following results.

**Lemma 2.4.** Suppose that \(0 < \lambda < \lambda_2.\) Then, for each \(u \in X_0\) with
\[\int_{\Omega} g(x)|u|^r dx > 0,\]
there exists a \(t_{b,\max} > 0\) such that,

(i) if \(\int_{\Omega} f(x) dx \leq 0,\)
then there is a unique \(t^- > t_{b,\max}\) such that \(t^- u \in N^{-}_\lambda\) and
\[J_\lambda(t^- u) = \sup_{t \geq 0} J_\lambda(tu);\]
(ii) if
\[ \int_{\Omega} f(x) \, dx > 0, \]
then there are unique \( t^+ \) and \( t^- \) with \( 0 < t^+ < t_{b,\text{max}} < t^- \) such that
\[ t^\pm u \in N_\lambda^\pm \quad \text{and} \quad J_\lambda(t^+ u) = \inf_{0 \leq t \leq t_{b,\text{max}}} J_\lambda(tu), \quad J_\lambda(t^- u) = \sup_{t \geq t_{b,\text{max}}} J_\lambda(tu). \]

**Proof.** Fix \( u \in X_0 \) with
\[ \int_{\Omega} g(x)|u|^{r} \, dx > 0. \]
Let
\[ h_b(t) = at^{p-q}\|u\|_{X_0}^p + bt^{(m+1)p-q}\|u\|_{X_0}^{(m+1)p} - t^{r-q} \int_{\Omega} g(x)|u|^{r} \, dx, \]
for \( a, t \geq 0 \). Clearly, \( tu \in N_\lambda \) if and only if
\[ h_b(t) = \lambda \int_{\Omega} f(x)|u|^q \, dx. \]
Since \( r > (m+1)p \) and
\[ \int_{\Omega} g(x)|u|^{r} \, dx > 0, \]
we have \( h_b(0) = 0, \)
\[ h(t) \longrightarrow -\infty \quad \text{as} \quad t \rightarrow \infty. \]
Moreover, there is a unique \( t_{b,\text{max}} > 0 \) such that \( h_b(t) \) achieves its maximum at \( t_{b,\text{max}} \), increasing for \( t \in [0, t_{b,\text{max}}) \) and decreasing for \( t \in (t_{b,\text{max}}, +\infty) \). In addition,
\[ t_{0,\text{max}} = \left( \frac{a(p-q)\|u\|_{X_0}^p}{(r-q) \int_{\Omega} g(x)|u|^{r} \, dx} \right)^{1/(r-p)}, \]
and
\[ h_0(t_{0,\text{max}}) = \left( \frac{a(p-q)\|u\|_{X_0}^p}{(r-q) \int_{\Omega} g(x)|u|^{r} \, dx} \right)^{(p-q)/(r-p)} a\|u\|_{X_0}^p. \]
SOLUTIONS FOR A KIRCHHOFF-TYPE PROBLEM

\[
- \left( \frac{a(p-q)\|u\|_{X_0}^p}{(r-q) \int_{\Omega} g(x) |u|^r \, dx} \right)^{(r-q)/(r-p)} \int_{\Omega} g(x) |u|^r \, dx
= \|u\|_{X_0}^p \frac{a(r-p)}{r-q} \left( \frac{a(p-q)\|u\|_{X_0}^r}{(r-q) \int_{\Omega} g(x) |u|^r \, dx} \right)^{(p-q)/(r-p)}
\geq \|u\|_{X_0}^p \frac{a(r-p)}{r-q} \left( \frac{a(p-q)}{(r-q)C_0^p \|g^+\|_\infty} \right)^{(p-q)/(r-p)}.
\]

(i)

\[
\int_{\Omega} f(x) \, dx \leq 0.
\]

There is a unique \( t^- > \tau_{b,\max} \) such that

\[
h_b(t^-) = \lambda \int_{\Omega} f(x) |u|^q \, dx
\]

and \( h_b'(t^-) < 0 \). Now,

\[
\psi_{t^-u}'(1) = t^- \psi_{u}'(t^-)
\]

(2.7)

\[
= M(\|t^-u\|_{X_0}^p)\|t^-u\|_{X_0}^p - \lambda \int_{\Omega} f(x)|t^-u|^q \, dx - \int_{\Omega} g(x)|t^-u|^r \, dx
= (t^-)^q \left( h_b(t^-) - \lambda \int_{\Omega} f(x)|u|^q \, dx \right)
= 0,
\]

and

(2.8)

\[
\psi_{t^-u}''(1) = (t^-)^2 \psi_{u}''(t^-) = (t^-)^{q+1} h_b'(t^-) < 0.
\]

Thus, \( t^- u \in N_\lambda^- \). Since \( t > \tau_{b,\max} \), we have \( h_b'(t) < 0 \). It follows from (2.7) and (2.8) that

\[
J_\lambda(t^-u) = \psi_u(t^-) = \sup_{t \geq 0} \psi_u(t) = \sup_{t \geq 0} J_\lambda(tu).
\]

(ii)

\[
\int_{\Omega} f(x) \, dx > 0.
\]
From (1.6) and (2.6), we have

\[ h_b(0) = 0 < \lambda \int_\Omega f(x)|u|^q dx \leq \lambda \|f^+\|_\infty C_0^q \|u\|_{X_0}^q \]
\[ \leq \|u\|_{X_0}^q \frac{a(r-p)}{r-q} \left( \frac{a(p-q)}{(r-q)C_0^r \|g^+\|_\infty} \right)^{(p-q)/(r-p)} \]
\[ \leq h_0(t_0, \max) < h_b(t_b, \max), \]

for \( \lambda \in (0, \lambda_2) \). Therefore, there are unique \( t^+ \) and \( t^- \) such that

\[ 0 < t^+ < t_{b, \max} < t^- \]

\[ h_b(t^+) = \lambda \int_\Omega f(x)|u|^q dx = h_b(t^-) \quad \text{and} \quad h_b'(t^+) > 0 > h_b'(t^-). \]

Similar to the argument in part (i), we conclude that \( t^+u \in \mathcal{N}_\lambda^+ \) and \( t^-u \in \mathcal{N}_\lambda^- \). Moreover,

\[ J_\lambda(t^-u) \geq J_\lambda(tu) \geq J_\lambda(t^+u) \quad \text{for each} \quad t \in [t^+, t^-] \]

and

\[ J_\lambda(t^+u) \leq J_\lambda(tu) \quad \text{for each} \quad t \in [0, t^+]. \]

Thus,

\[ J_\lambda(t^+u) = \inf_{0 \leq t \leq t_{b, \max}} J_\lambda(tu), \quad J_\lambda(t^-u) = \sup_{t \geq t_{b, \max}} J_\lambda(tu). \]

\[ \square \]

**Lemma 2.5.** Suppose that \( 0 < \lambda < \lambda_2 \). Then, for each \( u \in X_0 \) with

\[ \int_\Omega f(x)|u|^q dx > 0, \]

there exists a \( \tilde{t}_{b, \max} > 0 \) such that,

(i) if

\[ \int_\Omega g(x)|u|^r dx \leq 0, \]

then there is a unique \( 0 < t^+ < \tilde{t}_{b, \max} \) such that \( t^+u \in \mathcal{N}_\lambda^+ \) and

\[ J_\lambda(t^+u) = \inf_{t \geq 0} J_\lambda(tu); \]
(ii) if
\[ \int_{\Omega} g(x)|u|^r \, dx > 0, \]
then there are unique \( t^+ \) and \( t^- \) with \( 0 < t^+ < t_{b,\text{max}} < t^- \) such that
\( t^+ u \in \mathcal{N}^+_\lambda \) and
\[ J_\lambda(t^+ u) = \inf_{0 \leq t \leq t_{b,\text{max}}} J_\lambda(tu), \quad J_\lambda(t^- u) = \sup_{t \geq t_{b,\text{max}}} J_\lambda(tu). \]

Proof. Fix \( u \in X_0 \) with
\[ \int_{\Omega} f(x)|u|^q \, dx > 0. \]
Let
\[ \overline{h}_b(t) = at^{p-r}||u||^p_{X_0} + bt^{(m+1)p-r}||u||^{(m+1)p}_{X_0} - \lambda t^{q-r} \int_{\Omega} f(x)|u|^q \, dx, \]
for \( a, t \geq 0 \). Clearly,
\[ \overline{h}_b(t) \to -\infty \quad \text{as} \quad t \to 0^+ \]
and
\[ \overline{h}_b(t) \to 0 \quad \text{as} \quad t \to +\infty. \]
Moreover, there is a unique \( \tilde{t}_{b,\text{max}} > 0 \) such that \( \overline{h}_b(t) \) achieves its maximum at \( \tilde{t}_{b,\text{max}} \), increasing for \( t \in [0, \tilde{t}_{b,\text{max}}) \) and decreasing for \( t \in (\tilde{t}_{b,\text{max}}, +\infty) \). In addition,
\[ \tilde{t}_{0,\text{max}} = \left( \frac{(r-q)\lambda \int_{\Omega} f(x)|u|^q \, dx}{a(r-p)||u||^p_{X_0}} \right)^{1/(p-q)}, \]
and
\[ \overline{h}_0(\tilde{t}_{0,\text{max}}) = \left( \frac{a(r-p)||u||^p_{X_0}}{(r-q)\lambda \int_{\Omega} f(x)|u|^q \, dx} \right)^{(r-p)/(p-q)} a||u||^p_{X_0} \]
\[ - \left( \frac{a(r-p)||u||^p_{X_0}}{(r-q)\lambda \int_{\Omega} f(x)|u|^q \, dx} \right)^{(r-q)/(p-q)} \lambda \int_{\Omega} f(x)|u|^q \, dx \]
\[ = ||u||^r_{X_0} \frac{a(p-q)}{r-q} \left( \frac{a(r-p)||u||^q_{X_0}}{(r-q)\lambda \int_{\Omega} f(x)|u|^q \, dx} \right)^{(r-q)/(p-q)}. \]
\begin{align*}
\geq \|u\|_{X_0}^r \frac{a(p-q)}{r-q} \left( \frac{a(r-p)}{(r-q)\lambda C_0^q \|f^+\|_\infty} \right)^{(r-q)/(p-q)}.
\end{align*}

The results of Lemma 2.5 are obtained by repeating the same argument of Lemma 2.4.

**Lemma 2.6.** Suppose that $0 < \lambda < (q\lambda_2)/(2p)$. Then, we have

(i) $\alpha_+ < 0$,

(ii) $\alpha_- > d_0$ for some $d_0 > 0$.

**Proof.**

(i) Let $u \in N_\lambda^+$. We have $\psi''_u(1) > 0$. By (2.2),

$$(r-q) \int_{\Omega} g(x)|u|^r dx \leq a(p-q)\|u\|_{X_0}^p + ((m+1)p-q)b\|u\|_{X_0}^{(m+1)p}.$$ 

Therefore,

$$J_\lambda(u) = \frac{1}{p} \hat{M}(\|u\|_{X_0}^p) - \frac{\lambda}{q} \int_{\Omega} f(x)|u|^q dx - \frac{1}{r} \int_{\Omega} g(x)|u|^r dx$$

$$= \frac{1}{p} \hat{M}(\|u\|_{X_0}^p) - \frac{1}{q} M(\|u\|_{X_0}^p)\|u\|_{X_0}^p + \frac{r-q}{rq} \int_{\Omega} g(x)|u|^r dx$$

$$\leq \frac{a(q-p)}{pq} \|u\|_{X_0}^p + \frac{(q-(m+1)p)b}{(m+1)pq} \|u\|_{X_0}^{(m+1)p}$$

$$+ \frac{r-q}{rq} \left( a(p-q)\|u\|_{X_0}^p + ((m+1)p-q)b\|u\|_{X_0}^{(m+1)p} \right)$$

$$\leq -\frac{a(p-q)(r-p)}{pqr} \|u\|_{X_0}^p$$

$$- \frac{(m+1)p-q}{(m+1)pq} (r-(m+1)p)b\|u\|_{X_0}^{(m+1)p}$$

$$< 0.$$ 

(ii) Let $u \in N_\lambda^-$. We have $\psi''_u(1) < 0$. From (1.6) and (2.2),

$$(2.9) \quad a(p-q)\|u\|_{X_0}^p \leq (r-q) \int_{\Omega} g(x)|u|^r dx - ((m+1)p-q)b\|u\|_{X_0}^{(m+1)p}$$

$$\leq (r-q) \int_{\Omega} g(x)|u|^r dx$$
\leq (r - q)C_0^r \|g^+\|_\infty \|u(x)\|_{X_0}^q.

It follows from (2.9) that
\[ \|u\|_{X_0} \geq \left( \frac{a(p - q)}{C_0^r \|g^+\|_\infty (r - q)} \right)^{1/(r - p)}. \]

According to (1.6) and (2.1), we have
\[
\begin{align*}
J_\lambda(u) &= \frac{1}{p} \tilde{M}(\|u\|_{X_0}^p) - \frac{\lambda}{q} \int \Omega f(x)|u|^q dx - \frac{1}{r} \int \Omega g(x)|u|^r dx \\
&= \frac{1}{p} \tilde{M}(\|u\|_{X_0}^p) - \frac{\lambda}{r} M(\|u\|_{X_0}^p)\|u\|_{X_0}^p \\
&\quad - \frac{(r - q)\lambda}{rq} \int \Omega f(x)|u|^q dx \\
&\geq \frac{(r - p)a}{pr} \|u\|_{X_0}^p + \frac{(r - (m + 1)p)b}{(m + 1)pr} \|u\|_{X_0}^{(m+1)p} \\
&\quad - \frac{(r - q)\lambda}{rq} \|u\|_{X_0}^q \\
&\geq \|u\|_{X_0}^q \left( \frac{a(p - q)}{rp} \frac{1}{\|u\|_{X_0}^{r - q}} - \frac{(r - q)\lambda}{rq} \|f^+\|_\infty C_0^q \right) \\
&\geq \left( \frac{a(p - q)}{C_0^r \|g^+\|_\infty (r - q)} \right)^{q/(r-p)} \left( \frac{a(r - p)}{rp} \left( \frac{a(p - q)}{C_0^r \|g^+\|_\infty (r - q)} \right)^{(p-q)/(r-p)} \\
&\quad - \frac{(r - q)\lambda}{rq} \|f^+\|_\infty C_0^q \right).
\end{align*}
\]

Thus, if \( \lambda < (q\lambda_2)/(2p) \), then \( \alpha_\lambda^- > d_0 \) for some \( d_0 > 0 \). \( \square \)

3. **Proof of the main result.** In this section, we show the existence of minimizers in \( N_\lambda^+ \) and \( N_\lambda^- \) for \( \lambda > 0 \) small enough. Let
\[
\lambda^* = \min \left\{ \lambda_1, \frac{q}{2p}, \lambda_2 \right\}.
\]

We have the following results.
Proposition 3.1. If \( 0 < \lambda < \lambda^* \), then the functional \( J_\lambda \) has a minimizer \( u_0^+ \) in \( \mathcal{N}_\lambda^+ \) and satisfies:

(i) \( J_\lambda(u_0^+) = \alpha_\lambda^+ = \inf_{u \in \mathcal{N}_\lambda^+} J_\lambda(u) < 0 \),

(ii) \( u_0^+ \) is a nonzero solution of problem (1.1).

Proof. Since \( J_\lambda \) is bounded from below on \( \mathcal{N}_\lambda^+ \), there exists a minimizing sequence \( \{u_n\} \subset \mathcal{N}_\lambda^+ \) such that

\[
\lim_{n \to \infty} J_\lambda(u_n) = \alpha_\lambda^+.
\]

Thus, by Lemma (2.2), the sequence \( \{u_n\} \) is bounded in \( X_0 \). Then, there exists a \( u_0^+ \), up to a sequence, such that

\( u_n \rightharpoonup u_0^+ \) weakly in \( X_0 \) as \( n \to \infty \).

Moreover, by [23, Lemma 8],

\[
u_n \longrightarrow u_0^+ \text{ in } L^k(\mathbb{R}^N),\]

\[
u_n \longrightarrow u_0^+ \text{ almost everywhere in } \mathbb{R}^N\]

as \( n \to \infty \). For any \( 1 \leq k < p_*^* \), by [5, Theorem IV-9], there exists an \( l(x) \in L^k(\mathbb{R}^N) \) such that

\[|u_n(x)| \leq l(x) \text{ almost everywhere in } \mathbb{R}^N.\]

Therefore, by the dominated convergence theorem, we have that

\[
\lambda \int_\Omega f(x)|u_n|^q \, dx \longrightarrow \lambda \int_\Omega f(x)|u_0^+|^q \, dx,
\]

\[
\int_\Omega g(x)|u_n|^r \, dx \longrightarrow \int_\Omega g(x)|u_0^+|^r \, dx,
\]

as \( n \to \infty \). Now, on \( \mathcal{N}_\lambda \), we have

\[
\frac{\lambda(r-q)}{rq} \int_\Omega f(x)|u_n|^q \, dx = \frac{a(r-p)}{pr} \|u_n\|_{X_0}^p
\]

\[
+ \frac{[r-(m+1)p]b}{(m+1)pr} \|u_n\|_{X_0}^{(m+1)p} - J_\lambda(u_n).
\]
Letting $n \to \infty$, from Lemma 2.6, (3.1) and (3.2), we obtain

$$
\int_{\Omega} f(x)|u_0^+|^q \, dx > 0.
$$

By Lemma 2.5, there exists a $t_1 > 0$ such that

$$
t_1 u_0^+ \in N_{\lambda}^+.
$$

Next, we show that

$$
u_n \rightarrow u_0^+ \text{ strongly in } X_0.
$$

Suppose that this is not true. Then,

$$
\|u_0^+\|_{X_0} < \liminf_{n \to \infty} \|u_n\|_{X_0}.
$$

Thus, for $u_n \in N_{\lambda}^+$,

$$
\lim_{n \to \infty} \psi_{u_n}'(t_1) = \lim_{n \to \infty} \left( t_1^{p-1} M(t_1^p \|u_n\|_{X_0}^p) \|u_n\|_{X_0}^p \\
- \lambda t_1^{q-1} \int_{\Omega} f(x)|u_n|^q \, dx - t_1^{r-1} \int_{\Omega} g(x)|u_n|^r \, dx \right)
= \psi_{u_0}'(t_1) = 0,
$$

that is, $\psi_{u_n}'(t_1) > 0$ for $n$ large enough. Since $u_n = 1 \cdot u_n \in N_{\lambda}^+$, we see that $\psi_{u_n}'(t) < 0$ for $t \in (0, 1)$ and $\psi_{u_n}'(1) = 0$ for all $n$. Therefore, we must have $t_1 > 0$. On the other hand, $\psi_{u_n}(t)$ is decreasing on $(0, t_1)$; thus,

$$
J_\lambda(t_1 u_0^+) \leq J_\lambda(u_0^+) < \lim_{n \to \infty} J_\lambda(u_n) = \alpha_\lambda^+,
$$

a contradiction. Hence,

$$
u_n \rightarrow u_0^+ \text{ strongly in } X_0.
$$

This implies

$$
J_\lambda(u_n) \rightarrow J_\lambda(u_0^+) = \alpha_\lambda^+ \text{ as } n \to \infty.
$$
namely, \( u_0^+ \) is a minimizer of \( J_\lambda \) on \( \mathcal{N}_\lambda^+ \). Using Lemma 2.1, \( u_0^+ \) is a nonzero solution to problem (1.1).

**Proposition 3.2.** If \( 0 < \lambda < \lambda^* \), then the functional \( J_\lambda \) has a minimizer \( u^-_0 \) in \( \mathcal{N}_\lambda^- \) and satisfies:

(i) \( J_\lambda(u^-_0) = \alpha^-_\lambda = \inf_{u \in \mathcal{N}_\lambda^-} J_\lambda(u) > 0 \),

(ii) \( u^-_0 \) is a nonzero solution of problem (1.1).

**Proof.** \( J_\lambda \) is bounded from below such that

\[
\lim_{J \to \infty} J_\lambda(\bar{u}_n) = \alpha^-_\lambda.
\]

By the same argument given in the proof of Proposition 3.1, there exists a \( u^-_0 \), up to a sequence, such that

\[
\bar{u}_n \rightrightarrows u^-_0 \text{ weakly in } X_0,
\]

\[
\bar{u}_n \to u^-_0 \text{ in } L^k(\mathbb{R}^N)
\]

and

\[
\tilde{u}_n \to u^-_0 \text{ almost everywhere in } \mathbb{R}^N,
\]

as \( n \to \infty \), for any \( 1 \leq k \leq p_s^* \). Moreover,

\[
\lambda \int_\Omega f(x)|\bar{u}_n|^q dx \longrightarrow \lambda \int_\Omega f(x)|u^-_0|^q dx,
\]

\[
\int_\Omega g(x)|\bar{u}_n|^r dx \longrightarrow \int_\Omega g(x)|u^-_0|^r dx,
\]

as \( n \to \infty \). Similarly, we have

\[
\frac{r-q}{rq} \int_\Omega g(x)|\bar{u}_n|^r dx = J_\lambda(\bar{u}_n) + \frac{a(p-q)}{pq} \|\bar{u}_n\|^p_{\mathcal{N}^+} \frac{(m+1)p-q}{(m+1)pq} \|\bar{u}_n\|_{X_0^{(m+1)p}}^p.
\]

Letting \( n \to \infty \), from Lemma 2.6, (3.4) and (3.5), we obtain

\[
\int_\Omega g(x)|u^-_0|^r dx > 0.
\]

From Lemma 2.4, there exists a \( t_2 > 0 \) such that \( t_2 u^-_0 \in \mathcal{N}_\lambda^- \).
Next, we show that
\[ \tilde{u}_n \longrightarrow u_0^- \text{ strongly in } X_0. \]
Suppose that this is not true. Then,
\[ \|u_0^-\|_{X_0} < \liminf_{n \to \infty} \|\tilde{u}_n\|_{X_0}. \]
Thus, for \( \tilde{u}_n \in N^-_\lambda \), similar to the argument in the proof of Proposition 3.1, we have
\[ J_\lambda(t_2 u_0^-) < \liminf_{n \to \infty} J_\lambda(t_2 \tilde{u}_n) \leq \lim_{n \to \infty} J_\lambda(t_2 \tilde{u}_n) \leq \lim_{n \to \infty} J_\lambda(\tilde{u}_n) = \alpha^-_\lambda, \]
a contradiction. Hence, \( \tilde{u}_n \to u_0^- \) strongly in \( X_0 \). This implies
\[ J_\lambda(\tilde{u}_n) \longrightarrow J_\lambda(u_0^-) = \alpha^-_\lambda \quad \text{as } n \to \infty, \]
namely, \( u_0^- \) is a minimizer of \( J_\lambda \) on \( N^-_\lambda \). Using Lemma 2.1, \( u_0^- \) is a nonzero solution to problem (1.1).

Proof of Theorem 1.1. By Propositions 3.1 and 3.2, we obtain that problem (1.1) has two nonzero solutions \( u_0^+ \in N^+_\lambda \) and \( u_0^- \in N^-_\lambda \) in \( X_0 \).
Since
\[ J_\lambda(u_0^\pm) = J_\lambda(|u_0^\pm|) \quad \text{and} \quad |u_0^\pm| \in N^\pm_\lambda, \]
we may assume that \( u_0^\pm \) are nonnegative solutions of problem (1.1). The proof of Theorem 1.1 is complete.  \( \square \)

REFERENCES


**Guizhou Minzu University, School of Data Science and Information Engineering, Guiyang, P.R. China**

*Email address: gzymychuchangmu@sina.com*

**Guizhou Minzu University, School of Data Science and Information Engineering, Guiyang, P.R. China**

*Email address: 295284483@qq.com*

**Guizhou Minzu University, School of Data Science and Information Engineering, Guiyang, P.R. China**

*Email address: 136577802@qq.com*