THE IDEAL OF UNCONDITIONALLY $p$-COMPACT OPERATORS

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ABSTRACT. We investigate the ideal $\mathcal{K}_{up}$, $1 \leq p \leq \infty$, of unconditionally $p$-compact operators. We obtain the isometric identities $\mathcal{K}_{up} = \mathcal{K}_{up} \circ \mathcal{K}_{up}$, $\mathcal{K}_{up}^{\text{max}} = \mathcal{L}_{p}^{*}$, $\mathcal{K}_{up}^{\text{min}} = \mathcal{O}_{p^{*}}$ and $\mathcal{K}_{up} = \mathcal{N}_{\text{up}}^{\text{Qdual}}$ and prove that, if $X^*$ has the approximation property or $Y$ has the $\mathcal{K}_{up}$-approximation property, then $\mathcal{K}_{up}(X,Y)$ is isometrically equal to $\mathcal{K}_{up}^{\text{min}}(X,Y)$, and the dual space $\mathcal{K}_{up}(X,Y)^*$ is isometric to $(\mathcal{L}_{p}^{*})^{*}(X^*,Y^*)$. As a consequence, for every Banach space $X$, we obtain the isometric identities $\mathcal{K}_{up}^{\text{max}}(\ell_{1}(\Gamma),X) = \mathcal{L}_{p}^{*}(\ell_{1}(\Gamma),X)$, $\mathcal{K}_{up}^{\text{min}}(\ell_{1}(\Gamma),X) = \ell_{\infty}(\Gamma)\mathcal{O}_{p^{*}}X$ and $\mathcal{K}_{up}(\ell_{1}(\Gamma),X)^* = \mathcal{D}_{p}^{*}(\ell_{\infty}(\Gamma),X^*)$.

1. Introduction. The main notion of the paper stems from the criterion of compactness. Grothendieck [7] proved that a subset $K$ of a Banach space $X$ is relatively compact if and only if, for every $\varepsilon > 0$, there exists a null sequence $(x_{n})$ in $X$ such that

$$K \subset \left\{ \sum_{n=1}^{\infty} \alpha_{n}x_{n} : (\alpha_{n}) \in B_{\ell_{1}} \right\}$$

and $\sup_{n} \|x_{n}\| \leq \sup_{x \in K} \|x\| + \varepsilon$, where $B_{\ell_{1}}$ denotes the closed unit ball of $\ell_{1}$ and, in general, $B_{Z}$ denotes the closed unit ball of a Banach space $Z$. From this result, the operator norm of a compact operator

$$T : Y \longrightarrow X$$
can be determined via null sequences as follows:

\[(†) \quad \| T \| = \inf \left\{ \sup_n \| x_n \| : \| x_n \| \longrightarrow 0, \right\}
\]

\[T(B_Y) \subset \left\{ \sum_n \alpha_n x_n : (\alpha_n) \in B_{\ell_1} \right\} \).

That criterion of compactness was naturally extended by Sinha and Karn [15] as follows. For \(1 \leq p < \infty\) and a subset \(K\) of \(X\), \(K\) is said to be relatively \(p\)-compact if there exists an \((x_n) \in \ell_p(X)\) such that

\[K \subset p{\text{-co}}(\{x_n\}) := \left\{ \sum_n \alpha_n x_n : (\alpha_n) \in B_{\ell_p^*} \right\},\]

where \(1/p + 1/p^* = 1\), and \(\ell_p(X)\) is the Banach space with the norm \(\| \cdot \|_p\) of all \(X\)-valued absolutely \(p\)-summable sequences. A linear map \(T : Y \longrightarrow X\) is said to be \(p\)-compact if \(T(B_Y)\) is a relatively \(p\)-compact subset of \(X\). The collection of all \(p\)-compact operators from \(Y\) to \(X\) is denoted by \(K_p(Y, X)\). In view of (†), the same method may be used for measuring \(p\)-compact operators. Similarly, Delgado, Piñeiro and Serrano [4, 5] introduced an operator ideal norm on \(K_p\). The norm \(\| \cdot \|_{K_p}\), \(1 \leq p < \infty\), on the space \(K_p(Y, X)\) is defined by

\[\| T \|_{K_p} = \inf \{ \| (x_n) \|_p : (x_n) \in \ell_p(X), T(B_Y) \subset p{\text{-co}}(\{x_n\}) \} \]

Then \([K_p, \| \cdot \|_{K_p}]\) is a Banach operator ideal [5].

For \(1 \leq p \leq \infty\), the closed subspace \(\ell_p^u(X)\) of \(\ell_p^w(X)\), the Banach space with the norm \(\| \cdot \|_p^w\) of all \(X\)-valued weakly \(p\)-summable sequences, consists of sequences \((x_n)\) satisfying

\[\| (0, \ldots, 0, x_m, x_{m+1}, \ldots) \|_p^w \longrightarrow 0,\]
as \(m \to \infty\). Elements in \(\ell_p^u(X)\) are called unconditionally \(p\)-summable sequences [8]. We say that a subset \(K\) of \(X\) is relatively unconditionally \(p\)-compact (\(u\)-\(p\)-compact) if there exists an \((x_n) \in \ell_p^u(X)\) such that \(K \subset p{\text{-co}}(\{x_n\})\). Also, a linear map \(T : Y \rightarrow X\) is said to be \(u\)-\(p\)-compact if \(T(B_Y)\) is a relatively \(u\)-\(p\)-compact subset of \(X\). The collection of all \(u\)-\(p\)-compact operators from \(Y\) to \(X\) is denoted by
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$K_{up}(Y, X)$, and the norm $\| \cdot \|_{K_{up}}$ on $K_{up}(Y, X)$ is defined by

$$\|T\|_{K_{up}} = \inf \{ \| (x_n)\|_p^w : (x_n) \in \ell_p^w(X) \text{ and } T(B_Y) \subset p\text{-co}\{\{x_n\}\} \}.$$  

Then, the ideal $[K, \| \cdot \|]_{K_{up}}$ of compact operators is isometrically equal to $[K_{u\infty}, \| \cdot \|_{K_{u\infty}}]$, and $[K_{up}, \| \cdot \|_{K_{up}}]$, $1 \leq p < \infty$, is a Banach operator ideal $[8, \text{Theorem 2.1}].$

The main purpose of this paper is to establish some relationships among the ideals $[K_{up}, \| \cdot \|_{K_{up}}]$, some well-known operator ideals and tensor norms based on the investigation of the ideal $[K_p, \| \cdot \|_{K_p}]$ of Galicer, Lassalle and Turco $[6, 11].$

2. A factorization of $K_{up}$. The following lemma may be verified from a standard argument.

**Lemma 2.1.** Let $K$ be a collection of sequences of positive numbers. If

$$\sup_{(k_j) \in K} \sum_{j=1}^{\infty} k_j < \infty \quad \text{and} \quad \lim_{l \to \infty} \sup_{(k_j) \in K} \sum_{j \geq l} k_j = 0,$$

then, for every $\varepsilon > 0$, there exists a sequence $(b_j)$ of real numbers with $b_j \nearrow \infty$ and $b_j \geq 1$ for all $j$ such that

$$\sup_{(k_j) \in K} \sum_{j=1}^{\infty} k_j b_j \leq (1 + \varepsilon) \sup_{(k_j) \in K} \sum_{j=1}^{\infty} k_j \quad \text{and} \quad \lim_{l \to \infty} \sup_{(k_j) \in K} \sum_{j \geq l} k_j b_j = 0.$$

**Theorem 2.2.** Let $1 \leq p \leq \infty$. Then, $T \in K_{up}(X, Y)$ if and only if there exist a quotient space $Z$ of $\ell_p^\ast$ ($c_0$ if $p = 1$), $R \in K_{up}(X, Z)$ and $S \in K_{up}(Z, Y)$ such that $T = SR$. In this case,

$$\|T\|_{K_{up}} = \inf \|S\|_{K_{up}} \|R\|_{K_{up}},$$

where the infimum is taken over all such factorizations.

**Proof.** The “if” part is clear and, in this case,

$$\|T\|_{K_{up}} \leq \inf \|S\|_{K_{up}} \|R\|_{K_{up}}.$$  

Let $T \in K_{up}(X, Y)$, and let $\varepsilon > 0$ be given. The following proof is essentially due to $[15, \text{Theorem 3.2}], [2, \text{Theorem 3.1}]$ and $[6, \text{Proposition 2.9}]$. Choose $(y_n) \in \ell_p^w(Y)$ such that $T(B_X) \subset p\text{-co}(\{y_n\})$
and \( \|y_n\|_p^w \leq \|T\|_{\mathcal{K}_w}(1 + \varepsilon) \). Define the operators
\[
E_y : \ell_p^* \to Y
\]
by
\[
E_y \alpha = \sum_n \alpha_n y_n,
\]
and
\[
\widehat{E}_y : \ell_p^*/\ker(E_y) \to Y
\]
by \( \widehat{E}_y[\alpha] = E_y \alpha \). Now, for each \( x \in X \), there exists an \( \alpha \in \ell_p^* \) such that
\[
Tx = \sum_n \alpha_n y_n.
\]
Define the map
\[
T_y : X \to \ell_p^*/\ker(E_y) \quad \text{by} \quad T_y x = [\alpha].
\]
Then, it is easily seen that \( T_y \) is well defined, linear and \( \|T_y x\| \leq \|x\| \) for every \( x \in X \). It follows that \( T = \widehat{E}_y T_y \).

Now, by an application of Lemma 2.1, there exists a sequence \( (\beta_n) \) of positive numbers with \( \lim_{n \to \infty} \beta_n = 0 \) and \( \beta_n < 1 \) such that
\[
(z_n) := (y_n/\beta_n) \in \ell_p^u(Y) \quad \text{and} \quad \|y_n/\beta_n\|_p^w \leq \|y_n\|_p^w(1 + \varepsilon).
\]
Define the operators
\[
D_\beta : \ell_p^* \to \ell_p^* \quad \text{and} \quad E_z : \ell_p^* \to Y
\]
by
\[
D_\beta \alpha = (\alpha_n \beta_n) \quad \text{and} \quad E_z \alpha = \sum_n \alpha_n z_n,
\]
respectively, and the map
\[
\widehat{D}_\beta : \ell_p^*/\ker(E_y) \to \ell_p^*/\ker(E_z) \quad \text{by} \quad \widehat{D}_\beta([\alpha]) = [\beta_n \alpha_n].
\]
Then, we see that \( \widehat{D}_\beta \) is well defined and linear. Consider
\[
[x_1] := [\beta_1 e_1], \ldots, [x_n] := [\beta_n e_n], \ldots \in \ell_p^*/\ker(E_z).
\]
Then, it is easily verified that \((\{x_n\})_{n=1}^\infty \in \ell^u_p(\ell^*_p/\ker(E_y)), \|(\{x_n\})_n\|^w_p \leq 1\) and 
\[
\widehat{D}_\beta(B_{\ell^*_p/\ker(E_y)}) \subset \left\{ \sum_n \alpha_n x_n : (\alpha_n) \in B_{\ell^*_p} \right\}.
\]
Thus, \(\widehat{D}_\beta\) is \(u\)-\(p\)-compact, and \(\|\widehat{D}_\beta\|_{\mathcal{K}_{up}} \leq 1\). Define the map 
\[
\widehat{E}_y : \ell^*_p/\ker(E_y) \longrightarrow Y \quad \text{by} \quad \widehat{E}_y([\alpha]) = E_y \alpha.
\]
Recall that \((z_n) \in \ell^u_p(Y)\). Then, 
\[
\widehat{E}_y(B_{\ell^*_p/\ker(E_y)}) \subset \left\{ \sum_n \alpha_n z_n : (\alpha_n) \in B_{\ell^*_p} \right\}.
\]
Therefore, \(\widehat{E}_y\) is \(u\)-\(p\)-compact and \(\|\widehat{E}_y\|_{\mathcal{K}_{up}} \leq \|(y_n)\|^w_p (1 + \varepsilon)\). It follows that \(\widehat{E}_y = \widehat{E}_z \widehat{D}_\beta\).

Now, we have the following commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
T_y \downarrow & & \downarrow \widehat{E}_y \\
\ell^*_p/\ker(E_y) & \xrightarrow{\widehat{D}_\beta} & \ell^*_p/\ker(E_z),
\end{array}
\]
and we have 
\[
\inf \|S\|_{\mathcal{K}_{up}} \|R\|_{\mathcal{K}_{up}} \leq \|\widehat{E}_z\|_{\mathcal{K}_{up}} \|\widehat{D}_\beta T_y\|_{\mathcal{K}_{up}} \\
\leq \|(y_n)\|^w_p (1 + \varepsilon) \leq \|T\|_{\mathcal{K}_{up}} (1 + \varepsilon)^2.
\]
Since \(\varepsilon > 0\) is arbitrary, \(\inf \|S\|_{\mathcal{K}_{up}} \|R\|_{\mathcal{K}_{up}} \leq \|T\|_{\mathcal{K}_{up}}\). \(\square\)

From the proof of Theorem 2.2, we also obtain a factorization of \(\mathcal{K}_p\) via \(\mathcal{K}_{up}\).

**Theorem 2.3.** Let \(1 \leq p \leq \infty\). Then, 
\[
T \in \mathcal{K}_p(X, Y)
\]
if and only if there exist a quotient space \(Z\) of \(\ell^*_p\) (\(c_0\) if \(p = 1\)), 
\(R \in \mathcal{K}_{up}(X, Z)\) and \(S \in \mathcal{K}_p(Z, Y)\) such that \(T = SR\). In this case,
∥T∥_K_p = \inf ∥S∥_K_p ∥R∥_K_{up}, where the infimum is taken over all such factorizations.

**Corollary 2.4.** Let 1 ≤ p ≤ ∞. Then [K_{up}, ∥ ∥_K_{up}] = [K_{up} ◦ K_{up}, ∥ ∥_K_{up}] and [K_p, ∥ ∥_K_p] = [K_p ◦ K_{up}, ∥ ∥_K_{up}].

### 3. The maximal hull and minimal kernel of [K_{up}, ∥ ∥_K_{up}]

Given a Banach operator ideal [A, ∥ ∥_A], we denote by [A_{max}, ∥ ∥_A_{max}], [A_{min}, ∥ ∥_A_{min}], [A_{sur}, ∥ ∥_A_{sur}], [A_{inj}, ∥ ∥_A_{inj}], [A^*, ∥ ∥_A^*] and [A_{dual}, ∥ ∥_A_{dual}], the maximal hull, minimal kernel, surjective hull, injective hull, adjoint ideal and dual ideal, respectively. The definitions may be found in [3, 14].

A classical p-compact operator T ∈ K_p(X, Y), 1 ≤ p ≤ ∞, from X to Y, is represented as

\[ T = \sum_n x_n^* \otimes y_n, \quad (x_n^*) ∈ ℓ_p(X^*), \quad (y_n) ∈ ℓ_p(Y), \]

and its norm is

\[ ∥T∥_{K_p} := \inf ∥(x_n^*)∥_p^w ∥(y_n)∥_p^w, \]

where the infimum is taken over all such representations of T. Then, [K_p, ∥ ∥_K_p] is a Banach operator ideal, cf., [3, subsection 22.3] and [14, subsection 18.3].

From [3, Proposition 9.8] and [8, Lemma 3.2], we have:

**Proposition 3.1.** Let 1 ≤ p ≤ ∞. Then, [K_{up}, ∥ ∥_K_{up}] = [K_{up}^*, ∥ ∥_K_{up}^*].

From Proposition 3.1 and [3, Corollary 9.8], we have:

**Corollary 3.2.** Let 1 ≤ p ≤ ∞, and let Γ be a set. Then, for every Banach space X, K_{up}(ℓ_1(Γ), X) is isometrically equal to K_{p^*}(ℓ_1(Γ), X).

A Banach operator ideal [A, ∥ ∥_A] is said to be associated to a tensor norm α if the canonical map

\[ (A(M, N), ∥ ∥_A) → M^* ⊗_α N \]
is an isometry for all finite-dimensional normed spaces $M$ and $N$. We denote by $\alpha\backslash\alpha$, $\backslash\alpha$ and $\alpha\slash$, the left-injective associate, right-injective associate, left-projective associate and right-projective associate, respectively, of $\alpha$. See [3, subsections 20.6, 20.7] for the corresponding definitions.

The following is a crucial tensor norm in this paper. Let $u \in X \otimes Y$. For $1 \leq p \leq \infty$, define

$$w_p(u) = \inf \left\{ \| (x_j) \|_p^w \| (y_j) \|_p^{w^*} : u = \sum_{j=1}^{n} x_j \otimes y_j, \ n \in \mathbb{N} \right\}.$$ 

Then, $w_p$ is a finitely generated tensor norm, cf., [3, Section 12]. For the definition of accessibility of tensor norms, see [3, subsection 21.1].

**Proposition 3.3.** Let $1 \leq p \leq \infty$. Then, the ideal $[K_{up}, \| \cdot \|_{K_{up}}]$ is associated to the totally accessible tensor norm $/w_{p^*}$.

**Proof.** Since $w_{p^*}$ is accessible, cf., [3, Theorem 21.5 (1)], by the symmetric version of [3, Proposition 21.1 (2)] $/w_{p^*}$ is totally accessible.

Now, let $\alpha$ be a finitely generated tensor norm associated to $[K_{up}, \| \cdot \|_{K_{up}}]$. Then by Corollary 3.2, for every $n \in \mathbb{N}$ and every finite-dimensional normed space $N$, we have the following isometries:

$$\ell_\infty^n \otimes_{w_p} N \rightarrow \mathcal{K}_{p^*}^{w}(\ell_1^n, N) \rightarrow K_{up}(\ell_1^n, N) \rightarrow \ell_\infty^n \otimes_{\alpha} N.$$ 

Then, using the proof of [6, Theorem 3.3], the proof is complete. $\square$

**Corollary 3.4.** Let $1 \leq p \leq \infty$. Then $[K_{up}^{max}, \| \cdot \|_{K_{up}^{max}}]$, $[K_{up}, \| \cdot \|_{K_{up}}]$ and $[K_{up}^{min}, \| \cdot \|_{K_{up}^{min}}]$ are all totally accessible.

**Proof.** By Proposition 3.3, $[K_{up}^{max}, \| \cdot \|_{K_{up}^{max}}]$ is associated to $/w_{p^*}$. Hence, by Proposition 3.3 and [3, Proposition 21.3], $[K_{up}^{max}, \| \cdot \|_{K_{up}^{max}}]$ is totally accessible. The other parts follow from [3, Exercise 21.2 (b)]. $\square$

We denote the ideal of $p$-factorable operators by $[\mathcal{L}_p, \| \cdot \|_{\mathcal{L}_p}]$, cf., [3, Section 18] and [14, subsection 19.3].
Theorem 3.5. Let $1 \leq p \leq \infty$. Then $[K_{\text{up}}^{\text{max}}, \| \cdot \|_{K_{\text{up}}^{\text{max}}}] = [L_{p^*}^{\text{sur}}, \| \cdot \|_{L_{p^*}^{\text{sur}}}]$ and $K_{\text{up}}^{\text{min}}(X, Y)$ is isometric to $X^* \hat{\otimes}_{w_p^*} Y$ for all Banach spaces $X$ and $Y$.

Proof. Since $[L_{p^*}, \| \cdot \|_{L_{p^*}}]$ is associated to $w_{p^*}$, see [3, subsection 17.12], by [3, Theorem 20.11 (2)], $[L_{p^*}^{\text{sur}}, \| \cdot \|_{L_{p^*}^{\text{sur}}}]$ is associated to $/w_{p^*}$. By Proposition 3.3, we obtain the first part since the maximal ideal associated to a finitely generated tensor norm is unique. Due to the fact that $/w_{p^*}$ is totally accessible, the second part follows from [3, Corollary 22.2]. \hfill $\square$

From [3, Corollary 9.8 and the symmetric version of Corollary 20.7] and Theorem 3.5, we have:

Corollary 3.6. Let $1 \leq p \leq \infty$, and let $\Gamma$ be a set. Then, for every Banach space $X$, $K_{\text{up}}^{\text{max}}(\ell_1(\Gamma), X)$ is isometrically equal to $L_{p^*}(\ell_1(\Gamma), X)$ and $K_{\text{up}}^{\text{min}}(\ell_1(\Gamma), X)$ is isometric to $\ell_\infty(\Gamma) \hat{\otimes}_{w_p^*} X$.

A Banach space $X$ is said to have the approximation property (AP) if, for every compact subset $K$ of $X$ and every $\varepsilon > 0$, there exists a finite rank operator $S$ on $X$ such that $\sup_{x \in K} \| Sx - x \| \leq \varepsilon$. Grothendieck [7] proved that $X$ has the AP if and only if, for every Banach space $Y$, $K(Y, X) = \overline{F(Y, X)}^\| \cdot \|$. Based on this criterion, Oja [12] and Lassalle and Turco [10] introduced the notion of approximation property related to a Banach operator ideal $[\mathcal{A}, \| \cdot \|_{\mathcal{A}}]$, where the norm ideal $\| \cdot \|_{\mathcal{A}}$ is taken into account, namely, given a Banach operator ideal $[\mathcal{A}, \| \cdot \|_{\mathcal{A}}]$, a Banach space $X$ is said to have the $\mathcal{A}$-AP if, for every Banach space $Y$, $\mathcal{A}(Y, X) = \overline{F(Y, X)}^\| \cdot \|_{\mathcal{A}}$.

The $K_{\text{up}}$-AP was investigated in [9], and it was shown that, if $X$ has the AP, then, for every $1 \leq p \leq \infty$, $X$ has the $K_{\text{up}}$-AP.

For a definition and further information on accessibility of Banach operator ideals, see [3, subsection 21.2]. The next lemma follows immediately.
Lemma 3.7. Let $[A, \| \cdot \|_A]$ be a totally accessible Banach operator ideal. Then, for all Banach spaces $Y$ and $X$, $\|T\|_A = \|T\|_{A\text{min}}$ for every $T \in \mathcal{F}(Y, X)$.

Proposition 3.8. Let $[A, \| \cdot \|_A]$ be a totally accessible Banach operator ideal. Then, for all Banach spaces $Y$ and $X$, $A(Y, X) = \mathcal{F}(Y, X)$ if and only if $A(Y, X)$ is isometrically equal to $A_{\text{min}}(Y, X)$.

Proof. Suppose that $A(Y, X) = \mathcal{F}(Y, X)$, and let $T \in A(Y, X)$. Then, there exists a sequence $(T_n)$ in $\mathcal{F}(Y, X)$ such that $\lim_{n \to \infty} \|T_n - T\|_A = 0$. Then, by Lemma 3.7, $(T_n)$ is a Cauchy sequence in $(A_{\text{min}}(Y, X), \| \cdot \|_{A\text{min}})$. Thus, there exists an $R \in A_{\text{min}}(Y, X)$ such that $\lim_{n \to \infty} \|T_n - R\|_{A\text{min}} = 0$. Hence, $T = R \in A_{\text{min}}(Y, X)$ and

$$\|T\|_{A\text{min}} = \lim_{n \to \infty} \|T_n\|_{A\text{min}} = \lim_{n \to \infty} \|T_n\|_A = \|T\|_A.$$ 

From [3, Proposition 22.1 (2)], the converse is always true. 

Consequently, for a totally accessible Banach operator ideal $A$, $X$ has the $A$-AP if and only if, for every Banach space $Y$, $A(Y, X)$ is isometrically equal to $A_{\text{min}}(Y, X)$. Hence, from Corollary 3.4 and [3, Proposition 22.1 (3)], we have the following result which should be compared with [11, Proposition 3.4].

Corollary 3.9. Let $1 \leq p \leq \infty$. Then, a Banach space $X$ has the $K_{\text{up}}$-AP (respectively, $K_{\text{up}}^{\text{max}}$-AP) if and only if, for every Banach space $Y$, $K_{\text{up}}(Y, X)$ (respectively, $K_{\text{up}}^{\text{max}}(Y, X)$) is isometrically equal to $K_{\text{up}}(Y, X)$.

4. The dual space of $(K_{\text{up}}(X, Y), \| \cdot \|_{K_{\text{up}}})$.

Proposition 4.1. Let $1 \leq p \leq \infty$. If $X^*$ has the AP, then, for every Banach space $Y$, $K_{\text{up}}(X, Y)$ is isometrically equal to $K_{\text{up}}^{\text{min}}(X, Y)$.

Proof. Let $Y$ be a Banach space, and let $T \in K_{\text{up}}(X, Y)$. Then, by Corollary 2.4, there exist Banach space $Z$, $R \in K_{\text{up}}(X, Z)$ and $S \in K_{\text{up}}(Z, Y)$ such that $T = SR$. It is well known that $X^*$ has the
AP if and only if, for every Banach space \( Y \),
\[
\mathcal{K}(X, Y) = \overline{\mathcal{F}(X, Y)}^{∥·∥}.
\]
Thus, we see that \( T \in (\mathcal{K}_{\text{up}} \circ \overline{\mathcal{F}})(X, Y) \). By [3, Proposition 25.2 (2)] and Corollary 3.4, \( T \in \mathcal{K}_{\text{up}}^{\text{min}}(X, Y) \). Also, by [3, Proposition 22.1 (3) and Corollary 22.5], we have
\[
\|T\|_{\mathcal{K}_{\text{up}}^{\text{max}}} \leq \|T\|_{\mathcal{K}_{\text{up}}} \leq \|T\|_{(\mathcal{K}_{\text{up}}^{\text{max}})_{\text{min}}} = \|T\|_{\mathcal{K}_{\text{up}}^{\text{max}}}. \quad \square
\]

We denote by \( \alpha^t \) and \( \alpha' \), respectively, the transposed and dual tensor norm of a tensor norm \( \alpha \), see [3, subsection 15.2]. The adjoint tensor norm is defined by \( \alpha^* := (\alpha^t)' = (\alpha')^t \).

**Theorem 4.2.** Let \( 1 \leq p \leq \infty \). If \( X^* \) has the AP or \( Y \) has the \( \mathcal{K}_{\text{up}} \)-AP, then the dual space \( (\mathcal{K}_{\text{up}}(X, Y), \|·\|_{\mathcal{K}_{\text{up}}}^* \) is isometric to \( ((\mathcal{L}^{\text{inj}}_p)^*(X^*, Y^*), \|·\|_{(\mathcal{L}^{\text{inj}}_p)^*}) \). Moreover, we have:

(a) if \( X^* \) has the AP, then, for every \( T \in (\mathcal{L}^{\text{inj}}_p)^*(X^*, Y^*) \) and every \( S \in \mathcal{K}_{\text{up}}(X, Y) \),
\[
\langle T, S \rangle = \text{tr}_{X^*}(S^*T).
\]

(b) If \( Y^* \) has the AP, then, for every \( T \in (\mathcal{L}^{\text{inj}}_p)^*(X^*, Y^*) \) and every \( S \in \mathcal{K}_{\text{up}}(X, Y) \),
\[
\langle T, S \rangle = \text{tr}_{Y^*}(TS^*).
\]

**Proof.** By Corollary 3.9 and Proposition 4.1, \( \mathcal{K}_{\text{up}}(X, Y) \) is isometrically equal to \( \mathcal{K}_{\text{up}}^{\text{min}}(X, Y) \). In view of Theorem 3.5, the canonical map
\[
J_{/w_p^*} : X^* \hat{\otimes} /w_p^* Y \longrightarrow \mathcal{K}_{\text{up}}^{\text{min}}(X, Y)
\]
in [3, Theorem 22.2] is an isometry.

Now, by [3, Proposition 20.10] and its symmetric version,
\[
(/w_p^*)' = /w_p' = /w_p^* = (w_p \})^*.
\]
Since \( \mathcal{L}_p \) is associated to \( w_p \), by [3, Theorem 20.11 (1)] \( \mathcal{L}^{\text{inj}}_p \) is associated to \( w_p \). Thus, \( (\mathcal{L}^{\text{inj}}_p)^* \) is associated to \( (w_p \})^* = (/w_p^*)'. \) Hence, by [3,
Theorem 17.5], we have the following isometries.

\[(L^{\text{inj}}_p)^* (X^*, Y^*), \| \cdot \|_{(L^{\text{inj}}_p)^*} \rightarrow (X^{*\hat{\otimes}/w_p}, Y)^* \rightarrow (\mathcal{K}_{\text{up}}(X, Y), \| \cdot \|_{\mathcal{K}_{\text{up}}}^*).\]

(a) Let \( T \in (L^{\text{inj}}_p)^*(X^*, Y^*) \). Then, by an application of [3, Theorem 17.15] it may be verified that the map

\[ \text{id}_{X^{*\otimes T}} : X^{*\otimes (w_p)^t} Y^{**} \rightarrow X^{*\otimes \pi} X^{**} \]

is continuous and \((w_p)^t = /w_p^*\), where \( \pi \) is the projective tensor norm. Let

\[ \text{id}_{X^{*\otimes T}} : X^{*\otimes (w_p)^t} Y^{**} \rightarrow X^{*\otimes \pi} X^{**} \]

be the continuous extension of \( \text{id}_{X^{*\otimes T}} \). Let

\[ \Phi : (\mathcal{K}_{\text{up}}(X, Y), \| \cdot \|_{\mathcal{K}_{\text{up}}}) \rightarrow \mathcal{N}(X^*, X^*) \]

be the composition of the following maps, where \([\mathcal{N}, \| \cdot \|_{\mathcal{N}}]\) is the ideal of nuclear operators:

\[ \mathcal{K}_{\text{up}}(X, Y) = \mathcal{K}_{\text{up}}^{\text{min}}(X, Y) \xrightarrow{J^{-1}_{/w_p^*}} X^{*\otimes/w_p^*} Y \]

\[ = X^{*\otimes (w_p)^t} Y \xrightarrow{\text{id}_{X^{*\otimes T}}^{\otimes Y}} X^{*\otimes (w_p)^t} Y^{**} \xrightarrow{\text{id}_{X^{*\otimes T}}^{\otimes Y}} X^{*\otimes \pi} X^{**} \]

\[ \rightarrow X^{*\otimes \pi} X^{**} \xrightarrow{i_T} X^{**\otimes \pi} X^{*} \xrightarrow{J_T} \mathcal{N}(X^*, X^*). \]

Here, \( i_Y : Y \rightarrow Y^{**} \) is the canonical isometry and \( i_T \) is the transposed map. Since \( X^* \) has the AP, it is well known that the trace map

\[ \text{tr}_{X^*} : (\mathcal{N}(X^*, X^*), \| \cdot \|_{\mathcal{N}}) \rightarrow \mathbb{C} \]

is well defined and continuous. It may easily be verified that, for every \( R \in \mathcal{F}(X, Y) \),

\[ \Phi(R) = R^* T \]

and \( \langle T, R \rangle = \text{tr}_{X^*}(R^* T) \), and then it follows that \( \langle T, S \rangle = \text{tr}_{X^*}(S^* T) \) for every \( S \in \mathcal{K}_{\text{up}}(X, Y) \).

(b) Let \( T \in (L^{\text{inj}}_p)^*(X^*, Y^*) \). Then, by [3, Theorem 17.15], the map

\[ \text{id}_{Y^{**\otimes T}} : Y^{**\otimes w_p \setminus X^*} \rightarrow Y^{**\otimes \pi} Y^* \]
is continuous. Let 
\[ \text{id}_{Y^{**}} \otimes T : Y^{**} \widehat{\otimes}_{w_p} X^* \rightarrow Y^{**} \widehat{\otimes}_\pi Y^* \]
be the continuous extension of \( \text{id}_{Y^{**}} \otimes T \). Let 
\[ \Phi : (\mathcal{K}_{\text{up}}(X, Y), \| \cdot \|_{\mathcal{K}_{\text{up}}}) \rightarrow \mathcal{N}(Y^*, Y^*) \]
be the composition of the following maps:
\[ \mathcal{K}_{\text{up}}(X, Y) = \mathcal{K}_{\text{up}}^{\min}(X, Y) \xrightarrow{J^{-1}_{w_p^*}} X^* \widehat{\otimes}_{w_p^*} Y \xrightarrow{i_*} Y \widehat{\otimes}_{w_p} X^* \xrightarrow{i_Y \otimes \text{id}_{X^*}} Y^{**} \widehat{\otimes}_{w_p} X^* \xrightarrow{i_Y \otimes \text{id}_{X^*}} \mathcal{N}(Y^*, Y^*). \]

Since \( Y^* \) has the AP, the trace map 
\[ \text{tr}_{Y^*} : (\mathcal{N}(Y^*, Y^*), \| \cdot \|_\mathcal{N}) \rightarrow \mathbb{C} \]
is well defined and continuous. As in the proof of (a), the proof is complete. \( \square \)

We denote the ideal of \( p \)-dominated operators by \( \mathcal{D}_p \), cf., [3, Section 19] and [14, subsection 17.4].

**Corollary 4.3.** Let \( 1 \leq p \leq \infty \), and let \( \Gamma \) be a set. Then, for every Banach space \( X \), the dual space \( \mathcal{K}_{\text{up}}(\ell_1(\Gamma), X)^* \) is isometric to \( \mathcal{D}_p^*(\ell_\infty(\Gamma), X^*) \) and, for every \( T \in \mathcal{D}_p^*(\ell_\infty(\Gamma), X^*) \) and every \( S \in \mathcal{K}_{\text{up}}(\ell_1(\Gamma), X), \langle T, S \rangle = \text{tr}_{\ell_\infty(\Gamma)}(S^*T). \)

**Proof.** Recall that \( \ell_\infty(\Gamma) \) has the AP. From the symmetric version of [3, Corollary 20.7] and the fact that \( w_p' = w_p^* \), in the proof of Theorem 4.2, we have the following isometries.
\[ (\mathcal{L}_p^*(\ell_\infty(\Gamma), X^*), \| \cdot \|_{\mathcal{L}_p^*}) \rightarrow (\ell_\infty(\Gamma) \widehat{\otimes}_{w_p^*} X)^* \]
\[ \text{} \rightarrow (\mathcal{K}_{\text{up}}(\ell_1(\Gamma), X), \| \cdot \|_{\mathcal{K}_{\text{up}}})^*. \]

Since \( \mathcal{L}_p^* \) is isometrically equal to \( \mathcal{D}_p^* \), cf., [3, subsection 17.12], we complete the proof. \( \square \)

**5. The ideal of \( \mathcal{A} \)-compact operators.** Carl and Stephani [1] introduced the notion of compactness determined by operator ideals. Let \( [\mathcal{A}, \| \cdot \|_\mathcal{A}] \) be a Banach operator ideal. A subset \( K \) of a Banach
space $X$ is said to be relatively $\mathcal{A}$-\textit{compact} if there exist a Banach space $Z$, $U \in \mathcal{A}(Z, X)$ and a relatively compact subset $C$ of $Z$ such that $K \subset U(C)$. A linear map

$$R : Y \to X$$

is said to be $\mathcal{A}$-\textit{compact} if $R(B_Y)$ is a relatively $\mathcal{A}$-compact subset of $X$. We denote by $\mathcal{K}_\mathcal{A}(Y, X)$ the space of all $\mathcal{A}$-compact operators from $Y$ to $X$.

Lassalle and Turco [11] introduced a method of measuring the size of relatively $\mathcal{A}$-compact sets. For a relatively $\mathcal{A}$-compact subset $K$ of $X$, let

$$m_\mathcal{A}(K; X) := \inf \{ \|U\|_\mathcal{A} : U \in \mathcal{A}(Z, X), \text{relatively compact } C \subset B_Z, K \subset U(C) \},$$

and let $\|R\|_{\mathcal{K}_\mathcal{A}} := m_\mathcal{A}(R(B_Y); X)$ for $R \in \mathcal{K}_\mathcal{A}(Y, X)$. Then, $[\mathcal{K}_\mathcal{A}, \| \cdot \|_{\mathcal{K}_\mathcal{A}}]$ is a Banach operator ideal, see [11, Section 2].

The following lemma combines, with standard modifications, [11, Remark 1.3] and the proof of [6, Proposition 2.9]. We give a proof for the sake of completeness.

**Lemma 5.1.** Let $1 \leq p \leq \infty$. If $(x_n) \in \ell^u_p(X)$, then, for every $\varepsilon > 0$, there exist a relatively compact subset $M$ of $B_{\ell^u_p}$ and $T \in \mathfrak{A}_{\ell^u_p}(\ell^w_p, X)$ with

$$\|T\|_{\mathfrak{A}_{\ell^u_p}} \leq \|(x_n)\|_{\ell^w_p} + \varepsilon$$

such that

$$p\text{-co}(\{x_n\}) = T(M).$$

**Proof.** Let $\varepsilon > 0$ be given. By an application of Lemma 2.1, there exists a sequence $(\beta_n)$ with $\lim_{n \to \infty} \beta_n = \infty$ and $\beta_n > 1$ such that $(\beta_n x_n) \in \ell^u_p(X)$ and $\|(\beta_n x_n)\|_{\ell^w_p} \leq \|(x_n)\|_{\ell^w_p} + \varepsilon$. We see that the set

$$M := \left\{ \left( \frac{\alpha_n}{\beta_n} \right)_{n=1}^\infty : (\alpha_n) \in B_{\ell^u_p} \right\}$$

is a relatively compact subset of $B_{\ell^u_p}$ and

$$T := \sum_{n=1}^\infty e_n \otimes \beta_n x_n \in \mathfrak{A}_{\ell^u_p}(\ell^w_p, X),$$
where \((e_n)\) is the sequence of canonical unit vectors in \(\ell_p\). This yields

\[
p\text{-co}(\{x_n\}) = \left\{ \sum_{n=1}^{\infty} \gamma_n \beta_n x_n : (\gamma_n) \in M \right\} = T(M)
\]

and

\[
\|T\|_{\mathcal{K}_{p^*}} \leq \|(e_n)\|_{p^*} \|(\beta_n x_n)\|_p^w \leq \|(x_n)\|_p^w + \varepsilon.
\]

**Proposition 5.2.** Let \(K \subset X\), and let \(1 \leq p \leq \infty\). The following statements are equivalent.

(a) \(K\) is relatively \(u\)-\(p\)-compact.

(b) \(K\) is relatively \(\mathcal{K}_{p^*}\)-compact.

(c) \(K\) is relatively \(\mathcal{K}_{up}\)-compact. In this case,

\[
m_{\mathcal{K}_{up}}(K; X) = m_{\mathcal{K}_{p^*}}(K; X) = \inf\{\|(x_n)\|_p^w : K \subset p\text{-co}(\{x_n\}), (x_n) \in \ell_p^u(X)\}.
\]

**Proof.**

(a) \(\Rightarrow\) (b). Let \((x_n) \in \ell_p^u(X)\) be arbitrary such that \(K \subset p\text{-co}(\{x_n\})\). Let \(\varepsilon > 0\) be given. By Lemma 5.1, there exist a relatively compact subset \(M\) of \(B_{\ell_p}^*\) and \(T \in \mathcal{K}_{p^*}(\ell_p^*, X)\) with \(\|T\|_{\mathcal{K}_{p^*}} \leq \|(x_n)\|_p^w + \varepsilon\) such that

\[
K \subset p\text{-co}(\{x_n\}) = T(M).
\]

Thus, \(K\) is relatively \(\mathcal{K}_{p^*}\)-compact and

\[
m_{\mathcal{K}_{p^*}}(K; X) \leq \|T\|_{\mathcal{K}_{p^*}} \leq \|(x_n)\|_p^w + \varepsilon.
\]

Since \(\varepsilon > 0\) and \((x_n) \in \ell_p^u(X)\) are arbitrary, we have

\[
m_{\mathcal{K}_{p^*}}(K; X) \leq \inf\{\|(x_n)\|_p^w : K \subset p\text{-co}(\{x_n\}), (x_n) \in \ell_p^u(X)\}.
\]

(b) \(\Rightarrow\) (c). Simple verification shows that \(\mathcal{K}_{p^*} \subset \mathcal{K}_{up}\) and \(\|\cdot\|_{\mathcal{K}_{up}} \leq \|\cdot\|_{\mathcal{K}_{p^*}}\). Hence, (b) \(\Rightarrow\) (c) follows and \(m_{\mathcal{K}_{up}}(K; X) \leq m_{\mathcal{K}_{p^*}}(K; X)\).

(c) \(\Rightarrow\) (a). Let \(Y\) be a Banach space, and let \(T \in \mathcal{K}_{up}(Y, X)\) be such that \(M\) is a relatively compact subset of \(B_Y\) and \(K \subset T(M)\). Let \(\varepsilon > 0\) be given. Choose \((z_n) \in \ell_p^u(X)\) with \(\|(z_n)\|_p^w \leq \|T\|_{\mathcal{K}_{up}} + \varepsilon\) such that

\[
K \subset T(M) \subset T(B_Y) \subset p\text{-co}(\{z_n\}).
\]
Then, $K$ is relatively $u$-$p$-compact, and we have
\begin{align*}
\inf \{ \| (x_n) \|_p^w : K \subset p\text{-co}(\{x_n\}), (x_n) \in \ell_p^u(X) \} & \leq \| (z_n) \|_p^w \\
& \leq \| T \|_{\mathcal{K}_{up}} + \varepsilon.
\end{align*}
Since $\varepsilon > 0$ is arbitrary,
\[
\inf \{ \| (x_n) \|_p^w : K \subset p\text{-co}(\{x_n\}), (x_n) \in \ell_p^u(X) \} \leq \| T \|_{\mathcal{K}_{up}}.
\]

**Corollary 5.3.** Let $1 \leq p \leq \infty$. Then, we have
\[
[\mathcal{K}_{up}, \| \cdot \|_{\mathcal{K}_{up}}] = [\mathcal{K}_{\mathcal{K}_{up}^*, \mathcal{K}_{\mathcal{K}_{up}^*}}, \| \cdot \|_{\mathcal{K}_{\mathcal{K}_{up}^*, \mathcal{K}_{\mathcal{K}_{up}^*}}}].
\]

Proposition 5.2 and [11, Corollary 2.3] improve [8, Corollary 2.9] by the following proposition.

**Proposition 5.4.** Let $K \subset X$, and let $1 \leq p \leq \infty$. Then, $K$ is relatively $u$-$p$-compact if and only if $i_X(K)$ is relatively $u$-$p$-compact in $X^{**}$. In this case,
\[
\inf \{ \| (x_n) \|_p^w : i_X(K) \subset p\text{-co}(\{x_n^{**}\}), (x_n^{**}) \in \ell_p^u(X^{**}) \} = \inf \{ \| (x_n^{**}) \|_p^w : i_X(K) \subset p\text{-co}(\{x_n^{**}\}), (x_n^{**}) \in \ell_p^u(X^{**}) \}.
\]

Also, Proposition 5.2 and [11, Corollary 2.4] give the following:

**Corollary 5.5.** Let $1 \leq p \leq \infty$. Then, $T \in \mathcal{K}_{up}(X, Y)$ if and only if $T^{**} \in \mathcal{K}_{up}(X^{**}, Y^{**})$. In this case, $\| T \|_{\mathcal{K}_{up}} = \| T^{**} \|_{\mathcal{K}_{up}}$.

For $1 \leq p \leq \infty$, a linear map $T : X \to Y$ is called *quasi unconditionally $p$-nuclear* (quasi $u$-$p$-nuclear) [8] if there exists an $(x_n^*) \in \ell_p^u(X^*)$ such that $\| T x \| \leq \| (x_n^*(x)) \|_p$ for every $x \in X$. This notion originates from [13], namely, when the space $\ell_p^u(X^*)$ is replaced by the space $\ell_p(X^*)$, the linear map $T$ is called the *quasi $p$-nuclear* operator. We denote by $\mathcal{N}_{up}^Q(X, Y)$ the collection of all quasi $u$-$p$-nuclear operators from $X$ to $Y$. For $T \in \mathcal{N}_{up}^Q(X, Y)$, let
\[
\| T \|_{\mathcal{N}_{up}^Q} := \inf \| (x_n^*) \|_p^u,
\]
where the infimum is taken over all such inequalities. As in the proof of [13, Lemma 4], it may be shown that $[N_{up}^Q, \| \cdot \|_{N_{up}^Q}]$ is a Banach operator ideal.

**Theorem 5.6.** Let $1 \leq p \leq \infty$. Then, $T \in K_{up}(X, Y)$ if and only if $T^* \in N_{up}^Q(Y^*, X^*)$. In this case, $\|T\|_{K_{up}} = \|T^*\|_{N_{up}^Q}$.

**Proof.** According to [8, Theorem 2.4], we need only to check that $\|T\|_{K_{up}} \leq \|T^*\|_{N_{up}^Q}$. If $T \in K_{up}(X, Y)$, then, by Corollary 5.5 and [8, Theorem 2.3], we have

$$\|T\|_{K_{up}} = \|T^{**}\|_{K_{up}} = \|T^*\|_{N_{up}^Q}. \quad \Box$$

**Acknowledgments.** The author would like to express sincere gratitude to the referee for valuable comments.

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