ABSTRACT. Recently, Merca [4] found the recurrence relation for computing the partition function $p(n)$ which requires only the values of $p(k)$ for $k \leq n/2$. In this article, we find the recurrence relation to compute the bipartition function $p_{-2}(n)$ which requires only the values of $p_{-2}(k)$ for $k \leq n/2$. In addition, we also find recurrences for $p(n)$ and $q(n)$ (number of partitions of $n$ into distinct parts), relations connecting $p(n)$ and $q_o(n)$ (number of partitions of $n$ into distinct odd parts).

1. Introduction. A partition of a positive integer $n$ is a non-increasing sequence of positive integers whose sum is $n$. Let $p(n)$ denote the number of partitions of $n$, $p_{-2}(n)$ denote the number of bipartitions of $n$, $q(n)$ denote the number of partitions of $n$ into distinct parts and $q_o(n)$ denote the number of partitions of $n$ into distinct odd parts. Throughout the paper, we set

$$p(0) = p_{-2}(0) = q(0) = q_o(0) = 1$$

and

$$p(x) = p_{-2}(x) = q(x) = q_o(x) = 0 \quad \text{if } x < 0.$$

The generating functions for $p(n)$, $p_{-2}(n)$, $q(n)$ and $q_o(n)$ are

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q; q)_{\infty}}, \tag{1.1}$$

$$\sum_{n=0}^{\infty} p_{-2}(n)q^n = \frac{1}{(q; q)_2^{\infty}}, \tag{1.2}$$
\(1.3\) \[\sum_{n=0}^{\infty} q(n)q^n = \frac{1}{(q; q^2)_{\infty}} = (-q; q)_{\infty},\]

and

\(1.4\) \[\sum_{n=0}^{\infty} q_o(n)q^n = (-q; q^2)_{\infty},\]

where \(|q| < 1\) and \((a; q)_{\infty} = (1-a)(1-aq)(1-aq^2)\cdots\) is the \(q\)-shifted factorial.

Euler [2] invented generating function (1.1) which gives rise to a recurrence relation for \(p(n)\),

\(1.5\) \[\sum_{k=-\infty}^{\infty} (-1)^k p\left(n - \frac{k(3k-1)}{2}\right) = \delta_{0,n},\]

where \(\delta_{i,j}\) is the Kronecker delta. To compute partition function \(p(n)\) using (1.5) requires the values of \(p(k)\) with \(k \leq n-1\). Numerous mathematicians have given other recurrence relations for the partition function \(p(n)\). In 2004, Ewell [3] found two recurrence relations for \(p(n)\):

\(1.6\) \[p(n) = \sum_{k=0}^{\infty} p\left(\frac{n - k(k + 1)/2}{4}\right) + 2 \sum_{k=1}^{\infty} (-1)^{k-1} p(n - 2k^2)\]

and

\(1.7\) \[p(n) = \sum_{k=0}^{\infty} p\left(\frac{n - k(k + 1)/2}{2}\right)\]

\[+ \sum_{k=1}^{\infty} (-1)^{k-1} \{p(n - k(3k - 1)) + p(n - k(3k + 1))\},\]

which requires the values of \(p(k)\) with \(k \leq n - 2\) to compute \(p(n)\). Over the years, it has been a challenge for mathematicians to find the recurrence relation for \(p(n)\) that requires less number of values of \(p(k)\) with \(k < n\). In 2016, using Ramanujan’s theta function, Merca [4] found the most efficient recurrence relation

\(1.8\) \[p(n) = \sum_{k=0}^{[n/2]} \sum_{j=-\infty}^{\infty} p(k)p\left(\left\lfloor \frac{n}{2} \right\rfloor - k - j(4j - 2 + (-1)^{n})\right),\]

which requires only the values of \(p(k)\) with \(k \leq n/2\) to compute \(p(n)\).
Inspired by their relations, in this paper, we find the recurrence relation for bipartition function $p_{-2}(n)$ that requires only the values of $p_{-2}(k)$ with $k \leq n/2$. In addition, we also find recurrences for $p(n)$ and $q(n)$, the relation connecting $p(n)$ and $q_o(n)$.

Ramanujan’s theta functions and Jacobi’s identity play a key role in proving our main results. For $|q| < 1$, Ramanujan’s theta functions [1, page 36, entry 22] are defined as

$$
\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \sum_{n=-\infty}^{\infty} q^{n(2n+1)} = \frac{(q^2; q^2)^2}{(q; q)_{\infty}}
$$

and

$$
f(-q) = (q; q)_{\infty} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}.
$$

Lemma 1.1 (Jacobi’s identity). [1, page 39, entry 24]. We have

$$
(q; q)_{\infty}^3 = \sum_{m=0}^{\infty} (-1)^m (2m + 1) q^{m(m+1)/2}.
$$

Our main result is stated in the next theorem.

Theorem 1.2. For each integer $n \geq 0$,

$$
p_{-2}(n) = \sum_{i,j=-\infty}^{\infty} \sum_{k=0}^{\lfloor n/2 \rfloor} p_{-2}(k)
$$

\[
\times p_{-2} \left( \frac{n}{2} \right) - k - j(4j - 1) - i(4i - 2 + (-1)^n) \right) 
\]

+ \sum_{i,j=-\infty}^{\infty} \sum_{k=0}^{\lfloor n-1/2 \rfloor} p_{-2}(k)

\[
\times p_{-2} \left( \frac{n - 1}{2} \right) - k - j(4j - 3) - i(4i - 2 - (-1)^n) \right) .
\]
More explicitly, the above result may be written as

\[(1.13)\]
\[p_{-2}(2n) = \sum_{i,j=-\infty}^{\infty} \sum_{k=0}^{n} p_{-2}(k)p_{-2}(n-k-j(4j-1)-i(4i-1))\]
\[+ \sum_{i,j=-\infty}^{\infty} \sum_{k=0}^{n-1} p_{-2}(k)p_{-2}(n-1-k-j(4j-3)-i(4i-3))\]

and

\[(1.14)\]
\[p_{-2}(2n+1) = 2 \sum_{i,j=-\infty}^{\infty} \sum_{k=0}^{n} p_{-2}(k)p_{-2}(n-k-j(4j-3)-i(4i-1)).\]

**Example 1.3.** We see by Theorem 1.2 that the values of \(p_{-2}(n)\) for \(n \in \{0,1,2,3,4,5,6,7\}\) are:

\[p_0 = 1,\]
\[p_1 = 2p_0^2 = 2,\]
\[p_2 = p_0(p_0 + 2p_1) = 5,\]
\[p_3 = 2p_0(p_0 + 2p_1) = 10,\]
\[p_4 = 2p_0(p_0 + p_1 + p_2) + p_1^2 = 20,\]
\[p_5 = 4p_0(p_1 + p_2) + 2p_1^2 = 36,\]
\[p_6 = p_0(3p_0^2 + 4p_1 + 2p_2 + 2p_3) + p_1(p_1 + 2p_2) = 65,\]
\[p_7 = 2p_0(p_0 + 2p_2 + 2p_3) + 2p_1(p_1 + 2p_2) = 110,\]

where here, and throughout this example, we set \(p_{-2}(n) = p_n\). With the above values in hand, we can compute the values of \(p_{-2}(14)\) and \(p_{-2}(15)\), i.e.,

\[p_{-2}(14) = 2p_0(p_1 + 2p_2 + 3p_3 + 2p_4 + p_5 + p_6 + p_7)\]
\[+ 2p_1(p_1 + 3p_3 + 2p_4 + p_5 + p_6)\]
\[+ p_2(3p_2 + 4p_3 + 2p_4 + 2p_5) + p_3(p_3 + 2p_4) = 2665\]

and

\[p_{-2}(15) = 2p_0(p_0 + 2p_1 + 2p_2 + 2p_3 + 2p_4 + 2p_6 + 2p_7)\]
\[+ 2p_1(p_1 + 2p_2 + 2p_3 + 2p_5 + 2p_6)\]
\[+ 2p_2(p_2 + 2p_4 + 2p_5) + 2p_3(p_3 + 2p_4) = 3956.\]
2. Proof of Theorem 1.2. We write

\[
(2.1) \quad \frac{1}{(q; q)_\infty} = \frac{1}{(q; q)_\infty} = \frac{1}{(q^2; q^2)_\infty} \times \frac{(q^2; q^2)_\infty}{(q; q)_\infty}.
\]

Substituting (1.9) into (2.1), we obtain

\[
(2.2) \quad \frac{1}{(q; q)_\infty} = \frac{1}{(q^2; q^2)_\infty} \sum_{k=\infty}^{\infty} q^{k(2k+1)}.
\]

Replacing \(q\) by \(-q\) in equation (2.2), we find that

\[
(2.3) \quad \frac{1}{(-q; -q)_\infty} = \frac{1}{(q^2; q^2)_\infty} \sum_{k=\infty}^{\infty} (-1)^k q^{k(2k+1)}.
\]

Therefore, we can write

\[
(2.4) \quad \sum_{n=0}^{\infty} p(2n)q^{2n} = \frac{1}{2} \left( \frac{1}{(q; q)_\infty} + \frac{1}{(-q; -q)_\infty} \right)
\]

\[
= \frac{1}{2(q^2; q^2)_\infty} \sum_{k=\infty}^{\infty} (1 + (-1)^k) q^{k(2k+1)}
\]

\[
= \frac{1}{(q^2; q^2)_\infty} \sum_{k=\infty}^{\infty} q^{2k(4k+1)}
\]

\[
= \sum_{n=0}^{\infty} p_{-2}(n)q^{2n} \sum_{k=\infty}^{\infty} q^{2k(4k-1)},
\]

which is equivalent to

\[
(2.5) \quad \sum_{n=0}^{\infty} p(2n)q^n = \sum_{n=0}^{\infty} p_{-2}(n)q^n \sum_{k=\infty}^{\infty} q^{k(4k-1)}.
\]

Using the Cauchy product of two power series, we find that

\[
(2.6) \quad \sum_{n=0}^{\infty} p(2n)q^n = \sum_{n=0}^{\infty} \sum_{k=\infty}^{\infty} p_{-2}(n - k(4k - 1))q^n.
\]

Equating coefficients of \(q^n\), we obtain

\[
(2.7) \quad p(2n) = \sum_{k=\infty}^{\infty} p_{-2}(n - k(4k - 1)).
\]
In a similar fashion, considering
\[
\sum_{n=0}^{\infty} p(2n+1)q^{2n+1} = \frac{1}{2}\left( \frac{1}{(q;q)_\infty} - \frac{1}{(-q;-q)_\infty} \right),
\]
we derive the following expression of \( p(2n+1) \) in terms of \( p_{-2}(n) \):
\[
p(2n+1) = \sum_{k=-\infty}^{\infty} p_{-2}(n - k(4k - 3)).
\]

Now, we consider
\[
\frac{1}{(q;q)_\infty^2} = \frac{1}{(q^2;q^2)_\infty^2} \times \frac{1}{(q;q)_\infty} \times \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}.
\]

Using (1.1), (1.2), and (1.9) in (2.10), we find that
\[
\sum_{n=0}^{\infty} p_{-2}(n)q^n = \sum_{k=0}^{\infty} p_{-2}(k)q^{2k} \sum_{n=0}^{\infty} p(n)q^n \sum_{j=-\infty}^{\infty} q^{j(2j+1)}.
\]

Using the Cauchy product of power series, we have
\[
\sum_{n=0}^{\infty} p_{-2}(n)q^n = \sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} \sum_{k=0}^{\infty} p_{-2}(k)p(n - 2k - j(2j + 1))q^n.
\]

Equating coefficients of \( q^n \) on both sides of (2.11), we obtain
\[
p_{-2}(n) = \sum_{j=-\infty}^{\infty} \sum_{k=0}^{[n/2]} p_{-2}(k)p(n - 2k - j(2j + 1))
\]
\[
= \sum_{j=-\infty}^{\infty} \sum_{k=0}^{[n/2]} p_{-2}(k)p(n - 2k - j(2j - 1))
\]
\[
= \sum_{j=-\infty}^{\infty} \sum_{k=0}^{[n/2]} p_{-2}(k)p(n - 2k - 2j(4j - 1))
\]
\[
+ \sum_{j=-\infty}^{\infty} \sum_{k=0}^{[n/2]} p_{-2}(k)p(n - 2k - 2j(4j - 3) - 1).
\]
Replacing \( n \) by \( 2n \) and \( n \) by \( 2n + 1 \) in (2.12), we find that

\[
(2.13) \quad p_{-2}(2n) = \sum_{j=-\infty}^{\infty} \sum_{k=0}^{n} p_{-2}(k)p(2n - 2k - 2j(4j - 1))
+ \sum_{j=-\infty}^{\infty} \sum_{k=0}^{n} p_{-2}(k)p(2n - 2k - 2j(4j - 3) - 1)
\]

and

\[
(2.14) \quad p_{-2}(2n + 1) = \sum_{j=-\infty}^{\infty} \sum_{k=0}^{n} p_{-2}(k)p(2n + 1 - 2k - 2j(4j - 1))
+ \sum_{j=-\infty}^{\infty} \sum_{k=0}^{n} p_{-2}(k)p(2n - 2k - 2j(4j - 3)).
\]

Using (2.7) and (2.9) in (2.13) and (2.14), we arrive at (1.12).

3. New recurrences for \( p(n) \) and \( q(n) \).

**Theorem 3.1.** For each nonnegative integer \( n \), we have

\[
(3.1) \quad \sum_{k=0}^{\infty} (-1)^{k+n} (2k + 1) p(n - k(k + 1)) = \begin{cases} (-1)^{\ell+m} & \text{if } n = \ell(3\ell - 1)/2 + 2m(3m - 1), \\ 0 & \text{otherwise}, \end{cases}
\]

where \( \ell \) and \( m \) are integers.

**Theorem 3.2.** For each integer \( n \geq 0 \), we have

\[
(3.2) \quad \sum_{k=0}^{\infty} (-1)^{k} q \left( n - k\frac{3k - 1}{2} \right) = \begin{cases} (-1)^{\ell} & \text{if } n = \ell(3\ell - 1), \\ 0 & \text{otherwise}, \end{cases}
\]

where \( \ell \) is an integer.

**Proof of Theorem 3.1.** We have

\[
(-q; -q)_{\infty} = \frac{(q^2; q^2)_{\infty}^3}{(q; q)_{\infty}(q^4; q^4)_{\infty}},
\]
that is, 
\[
\frac{(q^2; q^2)_\infty^3}{(-q; -q)_\infty} = (q; q)_\infty (q^4; q^4)_\infty.
\]
Using (1.10) and (1.11) in the above equation, we obtain
\[
\sum_{n,k=0}^{\infty} (-1)^{k+n}(2k + 1)p(n)q^{n+k(k+1)}
= \sum_{\ell,m=-\infty}^{\infty} (-1)^{\ell+m}q^{\ell(3\ell-1)/2+2m(3m-1)}.
\]
Result (3.1) follows from (3.3) by extracting like powers of \(q\).

**Proof of Theorem 3.2.** We write
\[
(q; q)_\infty = (q; q^2)_\infty (q^2; q^2)_\infty,
\]
which is equivalent to
\[
\frac{1}{(q; q^2)_\infty}(q; q)_\infty = (q^2; q^2)_\infty.
\]
Substituting (1.3) and (1.10) into (3.4), we find that
\[
\sum_{n=0}^{\infty} q(n)q^n \sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k-1)/2} = \sum_{\ell=-\infty}^{\infty} (-1)^\ell q^{\ell(3\ell-1)},
\]
from which the result (3.2) follows.

**4. Relation connecting \(p(n)\) and \(q_o(n)\).**

**Theorem 4.1.** For each \(n \geq 0\),
\[
\sum_{k=-\infty}^{\infty} p\left(\left\lfloor \frac{n}{2} \right\rfloor - k(12k - 3 + (-1)^n2)\right)
- \sum_{k=-\infty}^{\infty} p\left(\left\lfloor \frac{n}{2} \right\rfloor - k(12k + 14 + (-1)^n3) - 4 - (-1)^n2\right) = q_o(n).
\]
Proof. Equation (1.10) can be expressed as

\[(4.2) \quad (q; q^2)_\infty = \frac{1}{(q^2; q^2)_\infty} \sum_{k=-\infty}^{\infty} (-1)^k q^{(3k-1)/2}.
\]

Replacing \(q\) by \(-q\) in (4.2), we obtain

\[(4.3) \quad (-q; q^2)_\infty = \frac{1}{(q^2; q^2)_\infty} \sum_{k=-\infty}^{\infty} (-1)^k (3k+1)/2 q^{(3k-1)/2}.
\]

However, we have

\[
\sum_{n=0}^{\infty} q_o(2n) q^{2n} = \frac{(q; q^2)_\infty + (-q; q^2)_\infty}{2} \]
\[
= \frac{1}{2(q^2; q^2)_\infty} \sum_{k=-\infty}^{\infty} ((-1)^k + (-1)^k (3k+1)/2) q^{(3k-1)/2}
\]
\[
= \frac{1}{(q^2; q^2)_\infty} \left( \sum_{k=-\infty}^{\infty} q^{2k(12k-1)} - \sum_{k=-\infty}^{\infty} q^{2k(12k+17)+12} \right),
\]

which is equivalent to

\[
\sum_{n=0}^{\infty} q_o(2n) q^n = \sum_{n=0}^{\infty} p(n) q^n \left( \sum_{k=-\infty}^{\infty} q^{k(12k-1)} - \sum_{k=-\infty}^{\infty} q^{k(12k+17)+6} \right).
\]

Using the Cauchy product of two power series, we find that

\[(4.4) \quad \sum_{n=0}^{\infty} q_o(2n) q^n = \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} p(n-k(12k-1)) q^n
\]
\[
- \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} p(n-k(12k+17) - 6) q^n.
\]

Equating coefficients of \(q^n\) on both sides of (4.4), we obtain

\[(4.5) \quad q_o(2n) = \sum_{k=-\infty}^{\infty} p(n-k(12k-1)) - \sum_{k=-\infty}^{\infty} p(n-k(12k+17) - 6).
\]
By taking
\[ \sum_{n=0}^{\infty} q_o(2n+1)q^{2n+1} = \frac{(-q;q^2)^{\infty} - (q;q^2)^{\infty}}{2}, \]
we also find in a similar fashion that
\[ (4.6) \quad q_o(2n+1) = \sum_{k=\infty}^{\infty} p(n-k(12k-5)) = \sum_{k=\infty}^{\infty} p(n-k(12k+11)-2). \]
Combining (4.5) and (4.6), we arrive at (4.1).

**Example 4.2.** If \( n = 23 \),
\[ p(11) - p(8) - p(9) + p(4) = 9, \]
and \( q_0(23) \) equals 9 since the nine partitions in question are:
\[
\begin{align*}
23, & \quad 19 + 3 + 1, \quad 17 + 5 + 1, \\
15 + 7 + 1, & \quad 15 + 5 + 3, \quad 13 + 9 + 1, \\
13 + 7 + 3, & \quad 11 + 9 + 3, \quad 11 + 7 + 5.
\end{align*}
\]

It would be interesting to find the recurrence relation for a \( t \)-tuple partition function denoted by \( p_{-t}(n) \), which would lead to a generalization of (1.8) and (1.12).

**Acknowledgments.** The authors would like to thank an anonymous referee for helpful comments. The first author would like to thank Prof. N.D. Baruah for information about the article [4] in a GIAN course at Tezpur University.

**REFERENCES**


**Bangalore University, Central College Campus, Department of Mathematics, Bengaluru-560 001, Karnataka, India and M.S. Ramaiah University of Applied Sciences, Department of Mathematics, Peenya Campus, #470-P, Peenya Industrial Area, Peenya 4th Phase, Bengaluru-560 058, Karnataka, India**

**Email address:** gireeshdap@gmail.com

**Bangalore University, Central College Campus, Department of Mathematics, Bengaluru-560 001, Karnataka, India**

**Email address:** msmnaika@rediffmail.com